USC High-School Math Competition - 2022
Written Test

1. Which of the following five numbers is the largest?

   (a) 1 (b) $\sqrt{2}$ (c) $\sqrt[3]{3}$ (d) $\sqrt[4]{4}$ (e) $\sqrt[5]{5}$

   Answer: (c)

2. Solve the equation: $||x-1|-|x||-|x-1|+|x|=1$.

   (a) $x=\pm\frac{3}{4}$ (b) $x=\pm\frac{1}{4}$ (c) $x=0$ (d) $x=\frac{1}{4}$ (e) $x=\frac{3}{4}$

   Answer: (e)

3. If $4^x-4^{x-1}=24$, the value of $(2x)^x$ equals:

   (a) $\sqrt{5}$ (b) $5\sqrt{5}$ (c) 25 (d) $25\sqrt{5}$ (e) 125

   Answer: (d)

4. For every positive integer n, the *Collatz function* $f(n)$ is defined by

   $$f(n)=\begin{cases}\frac{n}{2} \text{ if } n \text{ is even,}\\ 3n+1 \text{ if } n \text{ is odd.}\end{cases}$$

   If we write $f^k(n)=(f\circ f\cdots\circ f)(n)$ for the the funciton where we apply f k times in succession, what is $f^{22}(20)$?

   (a) 1 (b) 2 (c) 4 (d) 7 (e) 22

   Answer: (a)

5. According to researcher Max Donelan, as reported in the journal *Science*, a kangaroo's tail should also be counted as a fifth leg!

   If a group of people, dogs, and kangaroos has 8 heads, 5 tails, and 29 legs, how many kangaroos does it contain?

   (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

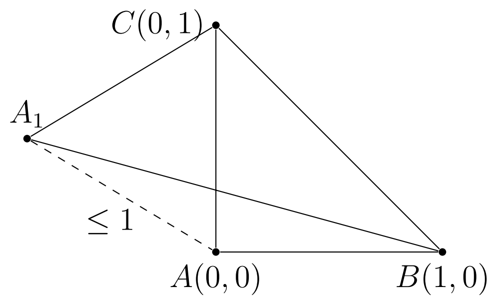
   Answer: (c)

6. Suppose $f(x)$ is a function which satisfies $f(0)=1$. Moreover, for any given real numbers x and y, we have $f(xy+1)=f(x)f(y)-f(y)-x+2$, then what is a formula for $f(x)$?

   (a) $1-x$ (b) $2x+1$ (c) x^2+1 (d) $x+1$ (e) $-x^2+2x+1$

   Answer: (d)

7. Let $A(0,0)$, $B(1,0)$, and $C(0,1)$ be three points in the Cartesian plane. If A_1 is a point in the plane such that the distance between A and A_1 does not exceed 1, then what is the largest possible area of $\triangle A_1BC$?



(a) $\frac{1}{2}$ (b) $\frac{\sqrt{2}}{2}$ (c) 1 (d) $\frac{1+\sqrt{2}}{2}$ (e) $\sqrt{2}$

Answer: (d)

8. If x^2+2x+5 is a factor of x^4+px^2+q, then the values of p and q are, respectively:

(a) 4 and 20 (b) 6 and 25 (c) 8 and 25 (d) 10 and 20 (e) 14 and 25

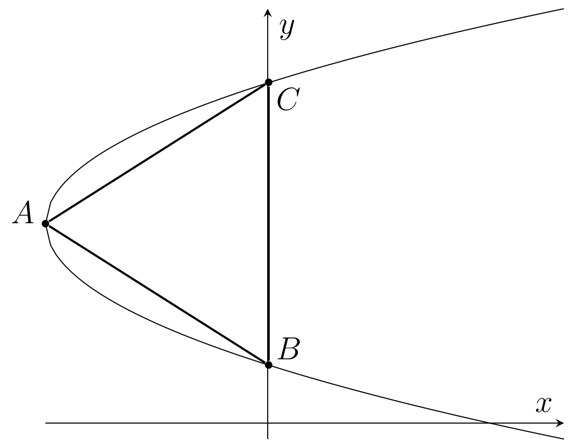
Answer: (b)

9. The solution set of the inequality $\sqrt{\log_2 x - 1} + \frac{1}{2}\log_{\frac{1}{2}} x^3 + 2 > 0$ is

(a) $[2,3)$ (b) $(2,3]$ (c) $[2,4)$ (d) $(2,4]$ (e) no solution for x

Answer: (c)

10. The graph of the quadratic function $x = y^2+2ay+\frac{a^2}{2}$ has the vertex A and intersects the y-axis at the points B and C. If ΔABC is an equilateral triangle, then what is the length of each side?



(a) $\sqrt{6}$ (b) $2\sqrt{2}$ (c) $2\sqrt{3}$ (d) $3\sqrt{2}$ (e) None of the above

Answer: (c)

11. How many roots of the equation $\cos 7x = \cos 5x$ in radians are in the interval $[0, \pi]$?

(a) 3 (b) 4 (c) 5 (d) 6 (e) 7

Answer: (e)

12. A function $F(n)$ satisfies $F(1) = F(2) = 1$ and $F(n) = F(n-1) + F(n-2)$ for every integer n.

What is $F(2022) + F(-2022)$?

(a) -2022 (b) 0 (c) 1 (d) 2021 (e) larger than 2021

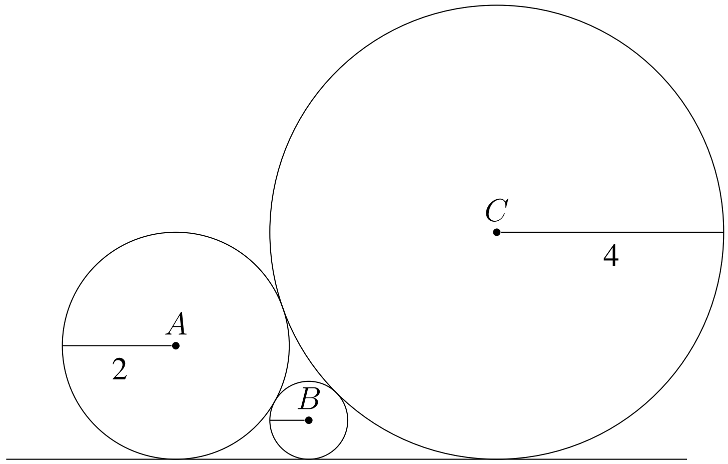
Answer: (b)

13. Suppose x and y are real numbers such that $x - \frac{1}{x} = y + \frac{1}{y}$. What is the value of $x^2 + \frac{1}{x^2} - \left(y^2 + \frac{1}{y^2}\right)$?

(a) -4 (b) 0 (c) 2 (d) 4 (e) It cannot be determined

Answer: (d)

14. Three circles on the same side of a common tangent line are exterior tangent to each other. Find the radius of the smallest circle if the larger circles have radii of 2 and 4, respectively.



(a) $\frac{1}{2}$ (b) $12 - 8\sqrt{2}$ (c) $\frac{1+\sqrt{2}}{3}$ (d) 1 (e) $\frac{3+\sqrt{2}}{4}$

Answer: (b)

15. Let n be the smallest integer for which you can write

$$n = a^2 + b^2 = c^2 + d^2,$$

where a, b, c, d are positive integers, all different from each other.

What is the sum of the digits of n?

(a) 8 (b) 9 (c) 10 (d) 11 (e) 12

Answer: (d) ($n = 65$)

16. In a board game, you roll two dice and your opponent rolls one. What is the probability the higher of your two rolls is more than your opponent's roll?

(a) $\frac{125}{216}$ (b) $\frac{23}{36}$ (c) $\frac{179}{216}$ (d) $\frac{181}{216}$ (e) $\frac{5}{6}$

Answer: (a)

17. Find the acute angles θ in radians such that the equation $x^2 + 4x\cos\theta + \cot\theta = 0$ has repeated roots.

(a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{5\pi}{12}$ (d) $\frac{\pi}{12}$ and $\frac{5\pi}{12}$ (e) $\frac{\pi}{6}$ and $\frac{5\pi}{12}$

Answer: (d)

18. Let a function $f(x)$ satisfy $f(x+1) = \dfrac{2}{1+\frac{1}{f(x)}}$, for any $x \geq 0$.

If $f(10) = \dfrac{2048}{2051}$, then what is $f(0)$?

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) $\frac{1}{2}$ (d) $\frac{4}{7}$ (e) 1

Answer: (b)

19. You play on a game show with the following rules. You are shown four prizes with different prices. You are given four price tags which give the prices of the prizes, but you don't know which tag goes with which prize.

You are asked to match the price tags to the prizes and, for each correct match, you win the prize. Unfortunately, you have no idea how much any of the prizes cost, and so you place the price tags randomly.

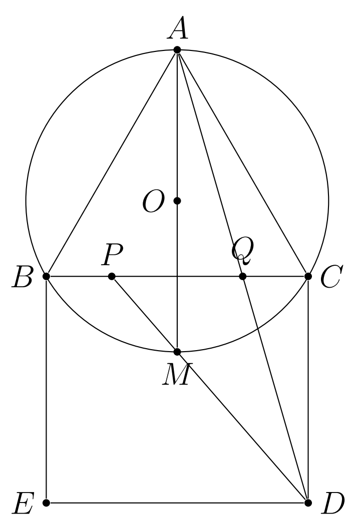
What is the most likely number of prizes you will win?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Answer: (a)

20. Let $\triangle ABC$ be an equilateral triangle and $[BCDE]$ be the rectangle with vertex D exterior to the circle *C* circumscribing $\triangle ABC$, such that, if AM is the diameter of the circle *C* and DM and AD intersect BC at P and Q, respectively, then $BP = QC$.

What is the ratio of the areas $\dfrac{S_{[BCDE]}}{S_{[ABC]}}$?



(a) $\frac{\sqrt{3}}{2}$ (b) 1 (c) $\frac{2\sqrt{3}}{3}$ (d) $\frac{4}{3}$ (e) 2

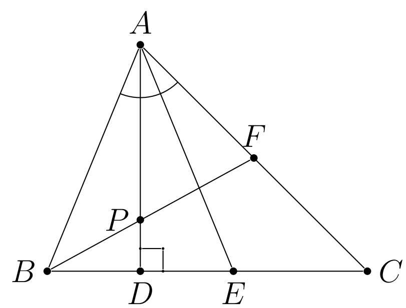
Answer: (e)

21. Let x and y be base 10 digits, such that the multiplication $27 \cdot 3y51 \;=\; 10x277$ is correct. What is the value of $x^2 + y^2$?

(a) 10 (b) 25 (c) 40 (d) 41 (e) 50

Answer: (e)

22. Let AD and AE trisect the angle $\angle BAC$ with points B, D, E, and C in this order, colinear, and such that $AD \perp BC$ and $AD = DC$. If AD intersects the median BF of the triangle $\triangle ABC$ at the point P, then find the ratio $\dfrac{AP}{PD}$,



(a) 2 (b) $2\sqrt{2}$ (c) $2+\sqrt{2}$ (d) $4-\sqrt{2}$ (e) 4

Answer: (c)

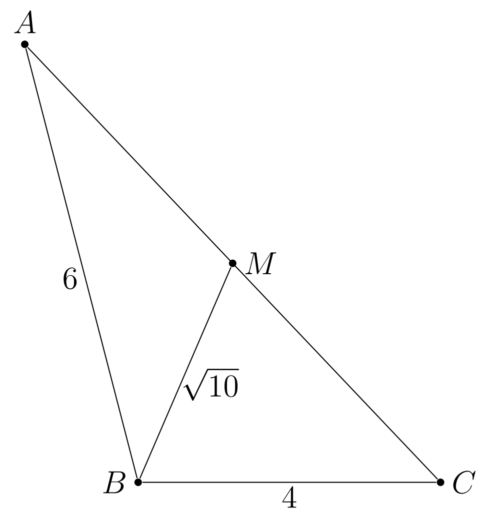
23. Let $n = 1111\cdots111$, with the digit 1 repeated 2022 times in total.

n is divisible by 3. What is the next smallest divisor of n?

(a) 7 (b) 9 (c) 11 (d) 13 (e) 100010001

Answer: (a)

24. In the triangle ΔABC, $AB = 6$, $BC = 4$. Let M be the middle point of AC and $BM = \sqrt{10}$. What is the value of $\sin^4\left(\frac{A}{2}\right) + \cos^4\left(\frac{A}{2}\right)$?

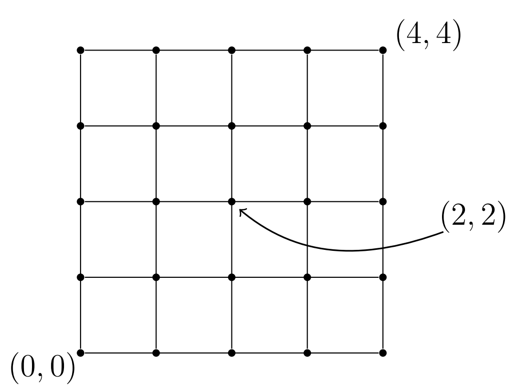


(a) $\frac{65}{81}$ (b) $\frac{2257}{2592}$ (c) $\frac{113}{128}$ (d) 1 (e) None of the above

Answer: (c)

25. A bug walks from the point $(0, 0)$ to $(4, 4)$ in eight steps – each a distance of 1 in either the positive x- or y-direction.

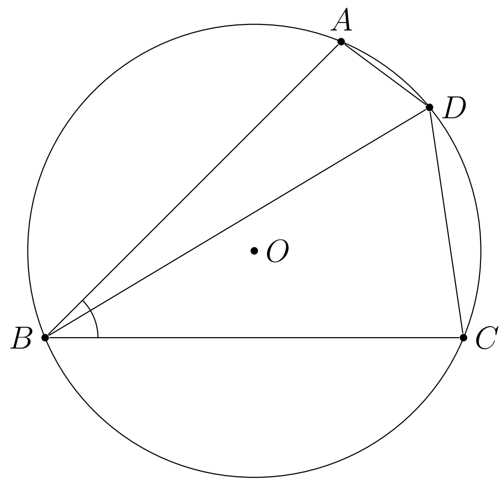
What is the probability that the bug passes through the point $(2, 2)$?



(a) $\frac{1}{5}$ (b) $\frac{16}{35}$ (c) $\frac{1}{2}$ (d) $\frac{18}{35}$ (e) $\frac{4}{5}$

Answer: (d)

26. Find the area of the inscriptible quadrilateral $[ABCD]$ with $AB = BC = 4\sqrt{2}$, angle $\angle ABC = 45^o$, and diagonal $BD = 6$.



(a) 12 (b) $9\sqrt{2}$ (c) $\dfrac{27}{2}$ (d) $6\sqrt{6}$ (e) 18

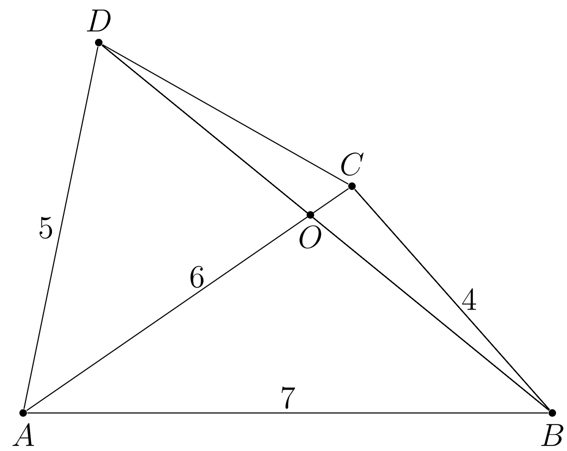
Answer: (b)

27. How many triples (x, y, z) satisfy the property that the product of any two of the real numbers x, y and z , after multiplied by 2021 and added to the third number, equals to 2022 ?

(a) 3 (b) 4 (c) 5 (d) 6 (e) None of the above

Answer: (c)

28. For the quadrilateral $ABCD$, let O be the intersection of AC and BD. If $\angle BAD + \angle ACB = 180°$, $BC = 4$, $AD = 5$, $AC = 6$ and $AB = 7$, then find $\dfrac{DO}{OB}$.



(a) $\dfrac{16}{15}$ (b) $\dfrac{15}{14}$ (c) $\dfrac{14}{13}$ (d) $\dfrac{13}{12}$ (e) None of the above

Answer: (b)

29. In the country of Mathlandia, money comes (only) in $8-$, $13-$, and $17-$cent denominations and dollar bills.

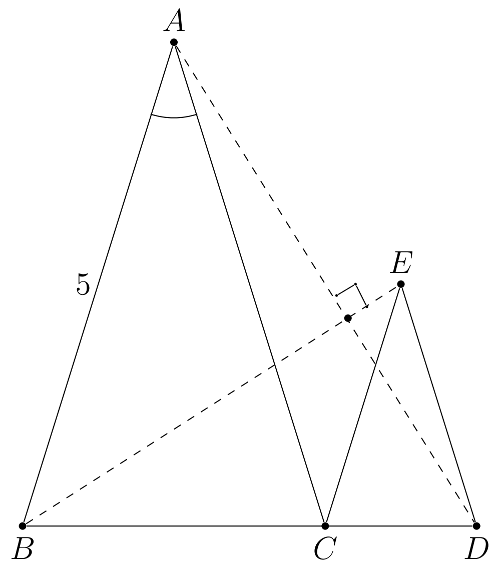
Janet enters a store, picks out a bag of candy, and hands a dollar to the cashier. “I’m sorry!”, the cashier replies, “I cannot make exact change for you. If you choose anything cheaper, I’d be able to.”

If n is the cost of the bag of candy (in cents), what is the sum of the digits of n?

(a) 8 (b) 9 (c) 10 (d) 11 (e) 12

Answer: (d)($n = 56$)

30. Let triangles $\triangle ABC$ and $\triangle ECD$ be similar isosceles triangles, sharing only the vertex C, with bases BC and CD colinear and vertices A and E on the same side of BD.
Assume that $AB = 5$ and that the angle $\angle BAC$ is maximum possible such that $AD \perp BE$.
What is the sum S of the areas of the two given triangles?



(a) 15 (b) 20 (c) 25 (d) 35 (e) 45

Answer: (a)