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Abstract

In the weighted space $L_{2,w}(B)$, where $w(x) = \pi^{-1}(1 - |x|^2)^{-1/2}$, B is the unit disk in $\mathbb{R}^2$, a complete orthogonal system of ridge polynomials is constructed. Ridge directions of the polynomials are condensed. Sufficient conditions for uniform convergence and uniform summability of Fourier series with respect to this system are found. A ridge polynomial basis for $C(B)$ is constructed.

A function $F(x \cdot \theta)$, where $x, \theta \in \mathbb{R}^2$, $x \cdot \theta$ is the inner product, F is an univariate function, is called a wave function (in x) with the wave direction θ. Finite linear combinations of wave functions are called ridge functions. Ridge approximation in L_2 was actively studied for last years by V.E. Majorov[1], K.I. Oskolkov[2, 3, 4], V.N. Temlyakov[5] and others. Many unexpected phenomena were found. For example, it turned out that the equidistributed ridge directions are not necessary optimal even for approximation of radial functions. A pretty Fourier analysis on the unit disk was constructed in [2], where the orthogonal basis consists of Chebyshev polynomial wave functions with equidistributed wave directions. Approximation properties of this basis in metrics different from L_2 was not known. Moreover, there were no any results on the ridge approximation in L_p, $p \neq 2$, in particular, for $p = \infty$. The goal of this paper is to find ridge expansions in C.

The following notations will be used throughout the paper:
$x \cdot y = x_1 y_1 + \ldots + x_d y_d$, $|x| = \sqrt{x \cdot x}$ for $x, y \in \mathbb{R}^d$, π_n^d is the space of polynomials in d variables of degree at most n, X_n denotes the n-th Legendre normalized polynomial, $S^d = \{x \in \mathbb{R}^{d+1} : |x| = 1\}$. If $f \in C(\Lambda)$, $\Lambda \subset \mathbb{R}^2$, then $\omega(f)$ denotes the modulus of continuity of f:

$$\omega(f, h) = \sup_{\substack{x, y \in \Lambda \\ |x-y| \leq h}} \|f(x) - f(y)\|_{C(\Lambda)}.$$

Let $B = \{x \in \mathbb{R}^2 : |x| \leq 1\}$, $w(x) = \pi^{-1}(1 - |x|^2)^{-1/2}$ for $x \in B$, we consider the weighted space $L_{2,w}$ of functions defined on B with the inner product $\langle f, g \rangle = \int_B fgw$.

We now present an orthogonal system in $L_{2,w}$ consisting of ridge polynomials. Set $\mathcal{P}_n = \operatorname{span}\{X_n(x \cdot \varphi), x \in B, \varphi \in S^1\}$ and prove that $\mathcal{P}_n$ is the orthogonal (in $L_{2,w}$) compliment to π_{n-1}^2 in π_n^2. It is proved in [2] that each polynomial in two variables can be represented as a linear combination of ridge polynomials of the same degree. By this fact, the orthogonality of $\mathcal{P}_n$ and π_{n-1}^2 follows from the following statement.

Lemma 1 *Let $Q \in \pi_{n-1}^1$, $\varphi, \psi \in S^1$, then*

$$\int_B Q(x \cdot \varphi) X_n(x \cdot \psi) w(x)\, dx = 0. \tag{1}$$

`Proof.` Without loss of generality one can consider $\psi = (1,0)$. The left hand side of (1) can be rewritten as

$$\frac{1}{\pi} \int_{-1}^{1} dx_1 X_n(x_1) \sum_{\ell + k < n} a_{\ell k}(\varphi) \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} x_1^k x_2^\ell \frac{dx_2}{\sqrt{1 - x_1^2 - x_2^2}}.$$

The internal integral is a polynomial in x_1 of degree $n-1$ because

$$\int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} x_1^k x_2^\ell \frac{dx_2}{\sqrt{1 - x_1^2 - x_2^2}} = \begin{cases} 0, & \text{if } \ell \text{ is odd,} \\ (1 - x_1^2)^{\ell/2} x_1^k \int_{-1}^{1} \frac{x_2^\ell\, dx_2}{\sqrt{1 - x_2^2}}, & \text{if } \ell \text{ is even.} \end{cases} \tag{2}$$

Since X_n is orthogonal to π_{n-1}^1, the external integral vanishes. $\diamond$

So, $\mathcal{P}_n$ is orthogonal to π_{n-1}^2. On the other hand, for each $p \in \pi_n^2$ we have

$$p(x) = \sum_{k=1}^{N} q_k(x \cdot \varphi_k) = \sum_{k=1}^{N} \sum_{\ell=0}^{n} a_{kl} X_\ell(x \cdot \varphi_k) = \sum_{k=1}^{N} a_{kn} X_n(x \cdot \varphi_k) + p'(x),$$

where $p' \in \pi_{n-1}^2$. Thus, $\mathcal{P}_n \oplus \pi_{n-1}^2 = \pi_n^2$ in $L_{2,w}$. The dimension of π_n^2 coincides with the number of monomials which equals $\frac{1}{2}(n+1)(n+2)$. Hence $\dim \mathcal{P}_n = n+1$. An orthogonal ridge basis can be constructed as follows. For an arbitrary φ_0 we set $P_0(x) = X_n(x \cdot \varphi_0)$. If $P_0, \ldots, P_k$ are already found and

$k < n$, then there exists $\varphi_{k+1} \in S^1$ such that $P_0(x), \dots, P_k(x), X_n(x \cdot \varphi_{k+1})$ are linear independent. Set

$$P_{k+1}(x) = A_0 P_0(x) + \ldots + A_k P_k(x) + X_n(x \cdot \varphi_{k+1}),$$

where

$$A_\ell = -\frac{1}{\|P_\ell\|_{2,w}^2} \int\limits_B P_\ell(x) X_n(x \cdot \varphi_{k+1}) w(x)\, dx.$$

It is clear, that $P_{k+1} \not\equiv 0$ and $\langle P_{k+1}, P_\ell \rangle = 0$.

To make this construction more explicit we discuss the choice of wave directions φ_k. The most natural way is to use equidistant nodes on S^1. Unfortunately, such φ_k are not suitable, the corresponding functions $X_n(x \cdot \varphi_k)$ are linear dependent. We will show that the equidistant nodes of a small enough arc are suitable.

Lemma 2 *For all* $\varphi, \psi \in S^1$

$$\int\limits_B X_n(x \cdot \varphi) X_n(x \cdot \psi) w(x)\, dx = \frac{X_n(\varphi \cdot \psi)}{X_n(1)}. \tag{3}$$

`Proof.` Without loss of generality we can consider $\psi = (1, 0)$. Define an operator Pr_{x_1} on B by

$$Pr_{x_1}(f) = \frac{1}{\pi} \int\limits_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} f(x) w(x)\, dx_2$$

and note that Pr_{x_1} takes any polynomial to a polynomial in one variable x_1 of the same degree. To check this fact one can consider a single monomial and use (2). In particular, if $f(x) = X_n(x \cdot \varphi)$, then

$$Pr_{x_1}(f) = \alpha(\varphi) X_n(x_1) + g(x_1), \quad g \in \pi_{n-1}^1. \tag{4}$$

Since, due to Lemma 1,

$$\int\limits_B Pr_{x_1}(f) g(x_1) w(x)\, dx =$$

$$\frac{1}{\pi^2}\int_{-1}^{1} dx_1 g(x_1) \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} X_n(x_1\varphi_1 + t\varphi_2)\frac{dt}{\sqrt{1-x_1^2-t^2}} \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \frac{dx_2}{\sqrt{1-|x|^2}} =$$

$$\int_B g(x_1)X_n(x\cdot\varphi)w(x)\,dx = 0,$$

and

$$\int_B X_n(x_1)g(x_1)w(x)\,dx = \int_{-1}^{1} X_n(x_1)g(x_1)\,dx_1 = 0,$$

we obtain $g\equiv 0$. Passing to the limit as $x_1\to 1$ in (4) we have

$$\alpha(\varphi) = \frac{X_n(\varphi_1)}{X_n(1)}.$$

To prove (3) it remains to note that

$$\int_B Pr_{x_1}(f)X_n(x_1)w(x)\,dx =$$

$$\frac{1}{\pi^2}\int_{-1}^{1} dx_1 X_n(x_1) \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} X_n(x_1\varphi_1 + t\varphi_2)\frac{dt}{\sqrt{1-x_1^2-t^2}} \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \frac{dx_2}{\sqrt{1-|x|^2}} =$$

$$\int_B X_n(x_1)X_n(x\cdot\varphi)w(x)\,dx$$

and

$$\int_B |X_n(x_1)|^2 w(x)\,dx = \int_{-1}^{1} |X_n(x_1)|^2\,dx_1 = 1. \quad \diamond$$

Let $m > 2n^2$, we set $\varphi_k = (\cos\frac{\pi k}{m}, \sin\frac{\pi k}{m})$ and prove that the functions $X_n(x\cdot\varphi_k)$, $k=0,\ldots,n$, are linear independent. Consider the Gram determinant for these functions. By Lemma 2, it looks as follows

$$\frac{1}{(X_n(1))^n}\begin{vmatrix} X_n(1) & X_n\left(\cos\frac{\pi}{m}\right) & \dots & X_n\left(\cos\frac{n\pi}{m}\right) \\ X_n\left(\cos\frac{\pi}{m}\right) & X_n(1) & \dots & X_n\left(\cos\frac{(n-1)\pi}{m}\right) \\ \vdots & \vdots & \ddots & \vdots \\ X_n\left(\cos\frac{n\pi}{m}\right) & X_n\left(\cos\frac{(n-1)\pi}{m}\right) & \dots & X_n(1) \end{vmatrix}.$$

It is known (see, e.g. [6], p.36) that this determinant equals

$$\left(\sum_{k=0}^{n} X_n\left(\cos\frac{k\pi}{m}\right)\right)\prod_{\ell=1}^{n}\left(X_n(1) - X_n\left(\cos\frac{\ell\pi}{m}\right)\right).$$

We should prove that this value is not equal to zero. It is well known that the Legendre polynomial takes its maximal value at the unique point $x = 1$. So, it is remains to verify that

$$\sum_{k=0}^{n} X_n\left(\cos\frac{k\pi}{m}\right) \neq 0. \tag{5}$$

Apply to each term of this sum the following formula (see, e.g. [7], p.116).

$$\begin{aligned} X_n(\cos\theta) &= 2\frac{(2n-1)!!}{(2n)!!}\cos n\theta + 2\frac{(2n-3)!!}{(2n-2)!!}\frac{1}{2}\cos(n-2)\theta + \\ & \quad 2\frac{(2n-5)!!}{(2n-4)!!}\frac{1}{2}\frac{3}{4}\cos(n-4)\theta + \dots, \end{aligned}$$

where the number of terms equals $[n/2]$. Since all the coefficients in this sum are positive and $\cos\frac{\ell k\pi}{m} > 0$, whenever $0 \le \ell \le n$, $0 \le k \le n$, the left hand side of (5) is positive.

So, we described a construction of orthogonal basis for $\mathcal{P}_n$. The elements of this basis are ridge polynomials with respect to $n+1$ wave directions. The directions are condensed as n goes to the infinity. An orthogonal basis for $\mathcal{P}_n$ is not unique. Another construction was presented in [8].

Let $\{P_{nk}\}_{k=0}^{n}$ be an orthonormal basis for $\mathcal{P}_n$. Then the entire collection $\{P_{nk}\}_{k,n}$ constitute an orthonormal polynomial system in $L_{2,w}$. Below it will be shown that this system is complete in $L_{2,w}$. For any appropriate function f we can consider the Fourier expansion with respect to $\{P_{nk}\}_{k,n}$:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\langle f, P_{nk}\rangle P_{nk}. \tag{6}$$

We are interested in convergence and summability of this series. Set

$$s_N(f) = \sum_{n=0}^{N}\sum_{k=0}^{n}\langle f, P_{nk}\rangle P_{nk}.$$

The convergence of (6) in the space $C(B)$ is closely related to the behavior of the Lebesgue constants

$$\mathcal{L}_N = \|s_N\|_{C(B)\to C(B)} = \max_{x\in B}\int\limits_B \left|\sum_{n=0}^{N}\sum_{k=0}^{n} P_{nk}(x)P_{nk}(t)\right| w(t)\,dt.$$

Similarly, a summability is related to the behavior of the Lebesgue constants of the corresponding linear means. We consider linear summation methods of type

$$s_N^{\Lambda}(f) = \sum_{n=0}^{N}\lambda_{Nn}\sum_{k=0}^{n}\langle f, P_{nk}\rangle P_{nk},$$

where $\Lambda = \{\lambda_{Nn}, N = 0, 1, \ldots,\ n = 0, \ldots, N\}$ is an infinite triangle matrix. The corresponding Lebesgue constant is

$$\mathcal{L}_N^{\Lambda} = \|s_N^{\Lambda}\|_{C(B)\to C(B)} = \max_{x\in B}\int\limits_B \left|\sum_{n=0}^{N}\lambda_{Nn}\sum_{k=0}^{n} P_{nk}(x)P_{nk}(t)\right| w(t)\,dt.$$

We now suggest a simple method for study the growth of Lebesgue constants for $s_N^{\Lambda}(f)$. For this we need some axelary tools.

Let $F \in C(S^2)$, the Laplace series of F is

$$\sigma(F, x) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\sqrt{n+\frac{1}{2}}\int\limits_{S^2} F(t)X_n(t\cdot x)\,ds(t),$$

where $x \in S^2$. We denote the partial sum of this series by

$$\sigma_N(F,x) = \frac{1}{2\pi}\sum_{n=0}^{N}\sqrt{n+\frac{1}{2}}\int_{S^2} F(t)X_n(t\cdot x)\,ds(t),$$

Note the following properties of these operators (see, e.g., [9], §6)

1. σ_N is a linear operator on $C(S^2)$;
2. σ_N takes $C(S^2)$ into π_N^3;
3. if $P \in \pi_N^3$, than $\sigma_N(P) = P$;
4. if $F \in C(S^2)$, $P \in \pi_N^3$, than

$$\int_{S^2} (F(t) - \sigma_N(F,t))P(t)\,ds(t) = 0.$$

For any $f \in C(B)$ we assign an associated function F defined on S^2 by: $F(x_1,x_2,x_3) = f(x_1,x_2)$ for all $x \in S^2$.

Theorem 3 *Let $f \in C(B)$, F be the function associated with f, then*

$$\sigma_N(F,x) = s_N(f,x_1,x_2) \tag{7}$$

for all $x \in S^2$.

`Proof.` First we show that the function $\sigma_N(F,x)$ does not depend on x_3. Indeed, for $x \in S^2$,

$$\sigma_N(F,x) = \frac{1}{2\pi}\sum_{n=0}^{N}\sqrt{n+\frac{1}{2}}\int_{S^2} F(t)X_n(x\cdot t)\,ds(t) =$$

$$\frac{1}{2\pi}\sum_{n=0}^{N}\sqrt{n+\frac{1}{2}}\left(\int_B f(t_1,t_2)X_n\left(x_1t_1+x_2t_2+x_3\sqrt{1-t_1^2-t_2^2}\right)\frac{dt}{\sqrt{1-t_1^2-t_2^2}}+\right.$$

$$\left.\int_B f(t_1,t_2)X_n\left(x_1t_1+x_2t_2-x_3\sqrt{1-t_1^2-t_2^2}\right)\frac{dt}{\sqrt{1-t_1^2-t_2^2}}\right).$$

It is clear that $\sigma_N(F,x)$ is a polynomial in x_1,x_2,x_3 without odd powers of x_3. If k is even, then $x_3^k = (1-x_1^2-x_2^2)^{k/2}$ that is a polynomial in x_1,x_2.

So $\sigma_N(F,x) = p(x_1,x_2)$, where $p \in \pi_N^2$. Let $q \in \pi_N^2$, P and Q denote the functions associated with p and q respectively. Using properties 3 and 4 of σ_N we have

$$\int\limits_B (f(x) - p(x))q(x)w(x)\,dx = \frac{1}{2\pi}\int\limits_{S^2}(F(x) - P(x))Q(x)\,ds(x) =$$

$$\frac{1}{2\pi}\int\limits_{S^2}(F(x) - \sigma_N(F,x))Q(x)\,ds(x) = 0$$

Thus, $\langle f - p, q\rangle = 0$ for all $q \in \pi_N^2$. In particularly, $\langle f, P_{nk}\rangle - \langle p, P_{nk}\rangle = 0$ for all $n = 0,\dots,N, k = 0,\dots,n$. But the functions P_{nk} constitute an orthonormal basis for π_N^2, hence

$$p = \sum_{k=0}^{N}\sum_{k=0}^{n}\langle p, P_{nk}\rangle P_{nk} = s_N(f). \ \diamond$$

Corollary 4 *Let $\Lambda = \{\lambda_{Nn}, N = 0,1,\dots,\ n = 0,\dots,N\}$, $f \in C(B)$ and let F be the function associated with f, then*

$$\frac{1}{2\pi}\sum_{n=0}^{N}\lambda_{Nn}\sqrt{n+\frac{1}{2}}\int\limits_{S^2}F(t)X_n(t\cdot x)\,ds(t) = s_N^{\Lambda}(f,x_1,x_2) \tag{8}$$

for any $x \in S^2$.

The proof follows immediately from (7), if we express both the left and the right hand sides of (8) by the Abel transform respectively via $\sigma_\ell(f)$ and $s_\ell(f)$, $l = 0,\dots,N$.

Theorem 5 *Let $\Lambda = \{\lambda_{Nn}, N = 0,1,\dots,\ n = 0,\dots,N\}$, then*

$$\|s_N^{\Lambda}\|_{C(B)\to C(B)} = \max_{x\in B}\int\limits_B\left|\sum_{n=0}^{N}\lambda_{Nn}\sum_{k=0}^{n}Pnk(t)Pnk(x)\right|w(t)\,dt \le$$

$$\int\limits_{-1}^{1}\left|\sum_{n=0}^{N}\lambda_{Nn}X_n(1)X_n(\tau)\right|\,d\tau. \tag{9}$$

`Proof.` Let F denote the function associated with f. By Corollary 4,

$$\|s_N^\Lambda(f)\|_{C(B)} \le \|\sigma_N^\Lambda(F)\|_{C(S^2)} \le \|F\|_{C(S^2)} \max_{x\in S^2} \int\limits_{S^2} \left| \sum_{n=0}^{N} \lambda_{Nn} \sqrt{n+\frac{1}{2}} X_n(t\cdot x) \right| ds(x). \quad (10)$$

We now prove that

$$\frac{1}{2\pi} \int\limits_{S^2} \left| \sum_{n=0}^{N} \lambda_{Nn} \sqrt{n+\frac{1}{2}} X_n(t\cdot x) \right| ds(t) = \int\limits_{-1}^{1} \left| \sum_{n=0}^{N} \lambda_{Nn} X_n(1) X_n(\tau) \right| d\tau$$

for any $x \in S^2$. Fix a point $x \in S^2$ and chose a Descartes coordinate system such that $x = (0,0,1)$. We have

$$\frac{1}{2\pi} \int\limits_{S^2} \left| \sum_{n=0}^{N} \lambda_{Nn} \sqrt{n+\frac{1}{2}} X_n(t_3) \right| ds(t) = \frac{1}{2\pi} \int\limits_{0}^{2\pi} d\varphi \int\limits_{0}^{\pi} \left| \sum_{n=0}^{N} \lambda_{Nn} X_n(1) X_n(\cos\theta) \right| \sin\theta \, d\theta = \int\limits_{-1}^{1} \left| \sum_{n=0}^{N} \lambda_{Nn} X_n(1) X_n(\tau) \right| d\tau.$$

¿From this and (10), taking into account that $\|F\|_{C(S^2)} = \|f\|_{C(B)}$, we obtain (9). $\diamond$

Theorem 6 *Let $f \in C(B)$, $x \in B$, then*

$$|f(x) - s_N(f,x)| \le A\sqrt{N}\omega\left(f, \frac{1}{N}\right), \quad (11)$$

where A is an absolute constant. In particular, if $f \in Lip\,\alpha$, $\alpha > 1/2$, then its Fourier series (6) uniformly converges to f on B.

`Proof.` For any $f \in C(B)$ there exists $F \in C([-1,1]^2)$ such that $F|_B = f$ and $\omega(f) = \omega(F)$ (see [10], Ch. IV, § 2). Let Q_N denote a polynomial of best approximation for F. By the multivariate Jackson theorem (see, e.g. [11], p. 293)

$$\|f - Q_N\| \leq C\omega\left(f, \frac{1}{N}\right).$$

This and Theorem 5 yield

$$\|f - s_N(f)\| = \|f - Q_N - s_N(f - Q_N)\| \leq$$
$$C\left(1 + \int\limits_{-1}^{1}\left|\sum_{k=0}^{N} X_k(1)X_k(t)\right| dt\right)\omega\left(f, \frac{1}{N}\right).$$

It remains to note that the integral in the right hand side is $O(\sqrt{n})$ (see, e.g. [13]). $\diamond$

Theorem 7 *Let φ be a continuous function on $[0,1]$, $\varphi(0) = 1$, $\varphi(1) = 0$,*

$$\int\limits_{0}^{\infty}\left|z\int\limits_{0}^{1}\varphi(x)xJ_0(zx)\,dx\right| dz < \infty,$$

where J_0 is the Bessel function. If $\Lambda = \{\varphi\left(\frac{n}{N}\right), N = 1,2,\dots,, n = 0,\dots,N\}$, then (6) is uniformly σ_N^Λ-summable for any $f \in C(B)$. In particularly, the uniform summability holds for the Hölder means $\varphi(u) = (1-u)^\alpha$ and for the Riesz-Böchner means $\varphi(u) = (1-u^2)^\alpha$ with $\alpha > 1/2$.

The proof immediately follows from Theorem 5 and a similar statement for the Legandre polynomials (see [12], Theorem 2).

Now it is clear that the system $\{P_{nk}, n = 1,2,\dots,k = 0,\dots,n,\}$ is complete in the space $C(B)$. Moreover it is complete in $L_{2,w}$.

Theorem 8 *Let $\alpha > 1/2$. For any $f \in C(B)$, its Fourier series (6) is uniformly (C,α)-summable on B.*

The proof immediately follows from Theorem 5 and a similar statement for the Legendre polynomials (see [14], p. 246). This theorem is not new. In fact, it was proved in [8] where another orthogononal basis for $\mathcal{P}_n$ was studied.

Both Theorem 7 and Theorem 8 imply the uniform Fejér summability. It follows that the Valle Poisson means

$$v_N(f,x) = \frac{1}{N}\sum_{\ell=N}^{2N-1}\sum_{n=0}^{\ell}\sum_{k=0}^{n}\langle f, P_{nk}\rangle P_{nk}$$

have uniformly bounded Lebesgue constants. The same arguments as that for Theorem 6 yield:

$$|f(x) - v_N(f,x)| \le C\omega\left(f, \frac{1}{N}\right), \tag{12}$$

where C is an absolute constant.

Finely, we construct a ridge polynomial basis for $C(B)$. We shall use the following approach based on the Peley-Wiener and the Krein-Milman-Rutman theorems (see [15] for details). Let $\{a_k\}_{k=0}^{\infty}$ be a basis for a Banach space H and let $f_k \in H^*$, $k = 0,1,\ldots$, be coefficient functionals for this basis. If $b_k \in H$ and

$$\|b_k - a_k\| \le \frac{2^{-k-2}}{\|f_k\|} =: \lambda_k$$

for all $k = 0,1,\ldots$, then the sequence $\{b_k\}_{k=0}^{\infty}$ is a basis for H.

We start with finding an initial basis $\{a_k\}$ for $C(B)$. In [16] it is constructed a polynomial basis $\{T_k\}_{k=0}^{\infty}$ for $C([-1,1]^2)$ with the following properties:

1. $\deg T_n \le C_1 n$, where C_1 is an absolute constant;

2. $T_n = t_\nu \otimes t_\mu$, where t_λ is a real polynomial in one variable, $\nu, \mu \le n$, $\|t_\lambda\|_\infty \le C_2\sqrt{\lambda}$, C_2 is an absolute constant;

3. $$\int_{[-1,1]^2} T_n(x)T_m(x)\frac{dx}{\sqrt{(1-x_1^2)(1-x_2^2)}} = \delta_{nm}.$$

Any function $f \in C\left([-1,1]^2\right)$ can be expanded with respect to this basis:

$$f(y) = \sum_{n=0}^{\infty}\int_{[-1,1]^2} f(x)T_n(x)\frac{dx}{\sqrt{(1-x_1^2)(1-x_2^2)}}T_n(y). \tag{13}$$

Define a map Φ on B by: if $x = (\rho\cos\varphi, \rho\sin\varphi)$, $0 \le \varphi < 2\pi$, $0 \le \rho \le 1$, then $\Phi(x) = (r(\varphi)\rho\cos\varphi, r(\varphi)\rho\sin\varphi)$, where $r(\varphi)$ is the length of the segment $\{x \in [-1,1]^2 : \ x_1 = \rho\cos\varphi, x_2 = \rho\sin\varphi\}$. It is clear that Φ takes B to $[-1,1]^2$ one to one, $\|f\|_{C([-1,1]^2)} = \|f(\Phi)\|_{C(B)}$ and there exists an absolute constant C_3 such that

$$\omega(f(\Phi), h) \le C_3\omega(f, h) \tag{14}$$

for all $f \in C\left([-1,1]^2\right)$. It follows from (13) that for each $f \in C(B)$ there exists a unique expansion with respect to the functions $a_n := T_n(\Phi)$. So, $\{a_n\}_{n=0}^{\infty}$ is a basis for $C(B)$ with the coefficient functionals defined by

$$f_n(f) = \int\limits_{[-1,1]^2} f\left(\Phi^{-1}(x)\right) T_n(x) \frac{dx}{\sqrt{(1-x_1^2)(1-x_2^2)}}.$$

Since

$$\|f_n\| = \int\limits_{[-1,1]^2} |T_n(x)| \frac{dx}{\sqrt{(1-x_1^2)(1-x_2^2)}} \le$$
$$\pi \int\limits_{[-1,1]^2} |T_n(x)|^2 \frac{dx}{\sqrt{(1-x_1^2)(1-x_2^2)}} = \pi,$$

we have $\lambda_N \ge \pi^{-1}2^{-n-2}$. Chose a sequence of positive integers $N_n \ge \gamma\, 2^n n^3$ where γ is a big enough constant, and set $b_n = V_{N_n}(a_n)$. By (12) and (14),

$$\|a_n - b_n\|_\infty \le C_4\omega\left(T_n, \frac{1}{N_n}\right).$$

Due to property 1 of T_n and the Markov inequality,

$$\omega\left(T_n, \frac{1}{N_n}\right) \le C_5\frac{n^2}{N_n}\|P_n\|_\infty.$$

Hence, taking into account that, by property 2, $\|P_n\|_\infty \le C_2^2 n$, we have

$$\|a_n - b_n\|_\infty \le \frac{C_2^2 C_4 C_5}{\gamma} 2^{-n}.$$

For $\gamma > 4\pi C_2^2 C_4 C_5$, this yields $\|a_n - b_n\| \leq \lambda_n$. Thus $\{b_n\}_{n=0}^{\infty}$ is a basis for $C(B)$.

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