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# Convergence of greedy approximation II. The triangular system

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# CONVERGENCE OF GREEDY APPROXIMATION II. THE TRIGONOMETRIC SYSTEM[1]

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Abstract. We study the following nonlinear method of approximation by trigonometric polynomials in this paper. For a periodic function $f$ we take as an approximant a trigonometric polynomial of the form $G_m(f) := \sum_{k\in\Lambda} \hat{f}(k)e^{i(k,x)}$, where $\Lambda \subset \mathbb{Z}^d$ is a set of cardinality $m$ containing the indices of the $m$ biggest (in absolute value) Fourier coefficients $\hat{f}(k)$ of function $f$. Note that $G_m(f)$ gives the best $m$-term approximant in the $L_2$-norm and, therefore, for each $f \in L_2$, $\|f - G_m(f)\|_2 \to 0$ as $m \to \infty$. It is known from previous results that in the case of $p \neq 2$ the condition $f \in L_p$ does not guarantee the convergence $\|f - G_m(f)\|_p \to 0$ as $m \to \infty$. We study the following question. What conditions (in addition to $f \in L_p$) provide the convergence $\|f - G_m(f)\|_p \to 0$ as $m \to \infty$? In the case $2 < p \leq \infty$ we find necessary and sufficient conditions on a decreasing sequence $\{A_n\}_{n=1}^{\infty}$ to guarantee the $L_p$-convergence of $\{G_m(f)\}$ for all $f \in L_p$, satisfying $a_n(f) \leq A_n$, where $\{a_n(f)\}$ is a decreasing rearrangement of absolute values of the Fourier coefficients of $f$.

## 1. Introduction

We study in this paper the following natural nonlinear method of summation of trigonometric Fourier series. Consider a periodic function $f \in L_p(\mathbb{T}^d)$, $1 \leq p \leq \infty$, $(L_\infty(\mathbb{T}^d) = C(\mathbb{T}^d))$, defined on the $d$-dimensional torus $\mathbb{T}^d$. Let a number $m \in \mathbb{N}$ and a number $t \in (0,1]$ be given and $\Lambda_m$ be a set of $k \in \mathbb{Z}^d$ with the properties:

$$\min_{k\in\Lambda_m} |\hat{f}(k)| \geq t \max_{k\notin\Lambda_m} |\hat{f}(k)|, \quad |\Lambda_m| = m, \tag{1.1}$$

where

$$\hat{f}(k) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(x)e^{-i(k,x)}dx$$

is a Fourier coefficient of $f$. We define

$$G^t_m(f) := G^t_m(f, \mathcal{T}) := S_{\Lambda_m}(f) := \sum_{k\in\Lambda_m} \hat{f}(k)e^{i(k,x)}$$

and call it an $m$-th weak greedy approximant of $f$ with regard to the trigonometric system $\mathcal{T} := \{e^{i(k,x)}\}_{k\in\mathbb{Z}^d}$. We write $G_m(f) = G^1_m(f)$ and call it an $m$-th greedy approximant.

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Clearly, an $m$-th weak greedy approximant and even an $m$-th greedy approximant may not be unique. In this paper we do not impose any extra restrictions on $\Lambda_m$ in addition to (1.1). Thus theorems formulated below hold for any choice of $\Lambda_m$ satisfying (1.1) or in other words for any realization $G^t_m(f)$ of the weak greedy approximation.

There has recently been (see surveys [D] and [T2]) much interest in approximation of functions by $m$-term approximants with regard to a basis (or minimal system). We will discuss in detail only results concerning the trigonometric system. T.W. Körner answering a question raised by Carleson and Coifman constructed in [K1] a function from $L_2(\mathbb{T})$ and then in [K2] a continuous function such that $\{G_m(f,\mathcal{T})\}$ diverges almost everywhere. It has been proved in [T1] for $p \neq 2$ and in [CF] for $p < 2$ that there exists a $f \in L_p(\mathbb{T})$ such that $\{G_m(f,\mathcal{T})\}$ does not converge in $L_p$. It was remarked in [T2] that the method from [T1] gives a little more: 1) There exists a continuous function $f$ such that $\{G_m(f,\mathcal{T})\}$ does not converge in $L_p(\mathbb{T})$ for any $p > 2$; 2) There exists a function $f$ that belongs to any $L_p(\mathbb{T})$, $p < 2$, such that $\{G_m(f,\mathcal{T})\}$ does not converge in measure. Thus the above negative results show that the condition $f \in L_p(\mathbb{T}^d)$, $p \neq 2$, does not guarantee convergence of $\{G_m(f,\mathcal{T})\}$ in the $L_p$-norm. The main goal of this paper is to find an additional (to $f \in L_p$) condition on $f$ to guarantee that $\|f - G_m(f,\mathcal{T})\|_p \to 0$ as $m \to \infty$. In Section 2 we prove the following theorem.

**Theorem 1.** *Let $f \in L_p(\mathbb{T}^d)$, $2 < p \leq \infty$, and let $q > p' := p/(p-1)$. Assume that $f$ satisfies the condition*

$$\sum_{|k|>n} |\hat{f}(k)|^q = o(n^{d(1-q/p')})$$

*where $|k| := \max_{1\leq j\leq d} |k_j|$. Then we have*

$$\lim_{m\to\infty} \|f - G^t_m(f,\mathcal{T})\|_p = 0.$$

For $f \in L_1(\mathbb{T}^d)$ let $\{\hat{f}(k(l))\}_{l=1}^\infty$ denote the decreasing rearrangement of $\{\hat{f}(k)\}_{k\in\mathbb{Z}^d}$, i.e.

$$|\hat{f}(k(1))| \geq |\hat{f}(k(2))| \geq \dots \quad . \tag{1.2}$$

Denote $a_n(f) := |\hat{f}(k(n))|$. In Section 3 we prove the following theorem.

**Theorem 2.** *Let $2 < p < \infty$ and let a decreasing sequence $\{A_n\}_{n=1}^\infty$ satisfy the condition:*

$$A_n = o(n^{1/p-1}) \quad as \quad n \to \infty. \tag{1.3}$$

*Then for any $f \in L_p(\mathbb{T}^d)$ with the property $a_n(f) \leq A_n$, $n = 1, 2, \dots,$ we have*

$$\lim_{m\to\infty} \|f - G^t_m(f,\mathcal{T})\|_p = 0. \tag{1.4}$$

We also prove in Section 3 that for any decreasing sequence $\{A_n\}$, satisfying

$$\limsup_{n\to\infty} A_n n^{1-1/p} > 0$$

there exists a function $f \in L_p$ such that $a_n(f) \leq A_n$, $n = 1, \dots,$ with divergent in the $L_p$ sequence of greedy approximants $\{G_m(f)\}$.

In Section 4 we prove a necessary and sufficient condition on the majorant $\{A_n\}$ to guarantee (under assumption that $f$ is a continuous function) uniform convergence of greedy approximants to a function $f$.

**Theorem 3.** *Let a decreasing sequence $\{A_n\}_{n=1}^\infty$ satisfy the condition $(\mathcal{A}_\infty)$:*

$$\sum_{M<n\le e^M} A_n = o(1) \quad as \quad M \to \infty. \tag{1.5}$$

*Then for any $f \in C(\mathbb{T})$ with the property $a_n(f) \le A_n$, $n = 1, 2, \dots$, we have*

$$\lim_{m\to\infty} \|f - G^t_m(f, \mathcal{T})\|_\infty = 0. \tag{1.6}$$

The condition $(\mathcal{A}_\infty)$ is very close to the convergence of the series $\sum_n A_n$; if it holds then we have

$$\sum_{n=1}^N A_n = o(\log_*(N)), \quad \text{as} \quad N \to \infty,$$

where a function $\log_*(u)$ is defined to be bounded for $u \le 0$ and to satisfy $\log_*(u) = \log_*(\log u) + 1$ for $u > 0$. The function $\log_*(u)$ grows slower than any iterated logarithmic function.

The condition $(\mathcal{A}_\infty)$ in Theorem 3 is sharp.

**Theorem 4.** *Assume that a decreasing sequence $\{A_n\}_{n=1}^\infty$ does not satisfy the condition $(\mathcal{A}_\infty)$. Then there exists a function $f \in C(\mathbb{T})$ with the property $a_n(f) \le A_n$, $n = 1, 2, \dots,$ and such that we have*

$$\limsup_{m\to\infty} \|f - G_m(f, \mathcal{T})\|_\infty > 0$$

*for some realization $G_m(f, \mathcal{T})$.*

Theorems 3 and 4 will be proved in Section 4. Also, in that section we will prove the following theorem.

**Theorem 5.** *Assume that a decreasing sequence $\{A_n\}_{n=1}^\infty$ is not summable. Then there exists a continuous function with the property $a_n(f) \le A_n$ such that its partial Fouries sums diverge at some point.*

We note (see Section 2) that sufficient conditions for convergence of greedy approximants in Theorem 1 for $p = \infty$ also imply the convergence of partial Fourier sums. Theorems 3 and 5 demonstrate that the conditions for convergence of greedy approximants in terms of decreasing rearrangement of Fourier coefficients of continuous functions are weaker than the ones for convergence of partial Fourier sums.

## 2. Sufficient conditions in terms of Fourier coefficients. Proof of Theorem 1

Let us begin this section with some historical remarks. The question of the rate of approximation of functions in certain smoothness classes by greedy approximants was discussed in [T1]. In particular the following function class was considered. For $0 < r < \infty$ and $0 < q \le \infty$, let $\mathcal{F}^r_q$ denote the class of those functions in $L_1(\mathbb{T}^d)$ such that

$$|f|_{\mathcal{F}^r_q} := \|(|k|^r|\hat{f}(k)|)_{k\in\mathbb{Z}^d}\|_{l_q} \le 1, \quad |\hat{f}(0)| \le 1.$$

Here we use the notation $|k| := \max\{|k_1|, \dots, |k_d|\}$. The following error estimates have been proved in [T1] for

$$G_m(\mathcal{F}^r_q)_p := \sup_{f\in \mathcal{F}^r_q} \|f - G_m(f)\|_p.$$

**Theorem 2.1.** *For any $0 < q < \infty$ and $r > d(1 - 1/q)_+$ we have*

$$G_m(\mathcal{F}^r_q)_p \asymp m^{-r/d-1/q+1/2}, \quad 1 \le p \le 2, \tag{2.1}$$

$$G_m(\mathcal{F}^r_q)_p \asymp m^{-r/d-1/q+1-1/p}, \quad 2 \le p \le \infty. \tag{2.2}$$

It has been also noticed in [T1] that the method used in the proof of Theorem 2.1 allows us to prove the order estimates similar to (2.1) and (2.2) for a little wider classes than $\mathcal{F}^r_q$. We define these classes now. It is easy to verify that for $f \in \mathcal{F}^r_q$ we have for each $l \ge 1$

$$\Big(\sum_{2^{l-1}\le|k|<2^l} |\hat f(k)|^q\Big)^{1/q} \le 2^{-r(l-1)}, \quad |\hat f(0)| \le 1. \tag{2.3}$$

We use the relation (2.3) as a definition of a new class $\mathcal{DF}^r_q$ ($\mathcal{D}$ stands here to stress that restrictions are imposed on the dyadic blocks). Here is a remark from [T1].

**Remark to Theorem 2.1.** *The relations (2.1) and (2.2) are valid when the class $\mathcal{F}^r_q$ is replaced by $\mathcal{DF}^r_q$.*

Denote for $r > 0$, $0 < q < \infty$, $\mathcal{F}o^r_q$ the space of functions $f \in L_1(\mathbb{T}^d)$ satisfying the condition

$$\sum_{|k|>n} |\hat f(k)|^q = o(n^{-rq}). \tag{2.4}$$

We will now prove Theorem 1 from Introduction.

**Theorem 1.** *Let $2 < p \le \infty$; $q > p' = p/(p-1)$. Assume that $f \in L_p(\mathbb{T}^d) \cap \mathcal{F}o^r_q$ with $r = d(1/p' - 1/q)$. Then for any $0 < t \le 1$ we have*

$$\|f - G^t_m(f)\|_p \to 0 \quad as \quad m \to \infty.$$

*Proof.* First we note that (2.4) is equivalent to

$$\sum_{k\in U(l)} |\hat f(k)|^q \le o(2^{-rlq}), \quad l = 1, 2, \dots, \tag{2.5}$$

where $U(l) := \{k \in \mathbb{Z}^d : 2^{l-1} \le |k| < 2^l\}$. It has been proved in [T1] (see relation (3.18)) that the estimates

$$\sum_{k\in U(l)} |\hat f(k)|^q \le 2^{-rlq}, \quad l = 1, 2, \dots,$$

imply

$$a_m(f) = O(m^{-r/d-1/q}).$$

In the same way one can prove that (2.5) implies that

$$a_m(f) = o(m^{-r/d-1/q}). \tag{2.6}$$

Taking into account that $r = d(1/p' - 1/q)$ we get from (2.6) that

$$a_m(f) = o(m^{-1/p'}).$$

In the case $2 < p < \infty$ we can finish the proof of Theorem 1 by applying Theorem 2 from Introduction. However, we choose to give an independent proof for the following two reasons. The proof below is simpler than the proof of Theorem 2 (see Section 3) and, also, the proof below covers the case $p = \infty$, where Theorem 2 does not hold (see Section 4).

Let

$$G_m^t(f) = S_{\Lambda_m}(f)$$

with $\Lambda_m$ satisfying (1.1). Consider first the case $2 < p < \infty$ and estimate $\|S_m^d(f) - S_{\Lambda_m}(f)\|_p$, where

$$S_m^d(f) := \sum_{k\in Q(m)} \hat{f}(k)e^{i(k,x)}, \qquad Q(m) := \{k : |k| \le m^{1/d}\}.$$

Then we have

$$S_m^d(f) - S_{\Lambda_m}(f) = \sum_{k\in Q(m)\setminus\Lambda_m} \hat{f}(k)e^{i(k,x)} - \sum_{k\in\Lambda_m\setminus Q(m)} \hat{f}(k)e^{i(k,x)} =: \Sigma_1 - \Sigma_2. \tag{2.7}$$

From the definition of $\Lambda_m$ we get

$$a_{m+1}(f) \le \max_{k\notin\Lambda_m} |\hat{f}(k)| \le t^{-1} \min_{k\in\Lambda_m} |\hat{f}(k)| \le t^{-1}a_m(f). \tag{2.8}$$

Thus by the Hausdorff-Young theorem (see [Z,Chap.12,Section 2]) we get

$$\|\Sigma_1\|_p \le (\sum_{k\in Q(m)\setminus\Lambda_m} |\hat{f}(k)|^{p'})^{1/p'} = O(a_m(f)m^{1/p'}) = o(1).$$

Using the Hausdorff-Young theorem again and using the Hölder inequality with a parameter $q/p'$ we get

$$\|\Sigma_2\|_p \le (\sum_{k\in\Lambda_m\setminus Q(m)} |\hat{f}(k)|^{p'})^{1/p'} \le (\sum_{k\in\Lambda_m\setminus Q(m)} |\hat{f}(k)|^{q})^{1/q} m^{1/p'-1/q} \le$$

$$\le (\sum_{k\notin Q(m)} |\hat{f}(k)|^{q})^{1/q} m^{1/p'-1/q} = o(1). \tag{2.9}$$

It remains to remark that $\|f - S_m^d(f)\|_p \to 0$ as $m \to \infty$.

Let us now consider the case $p = \infty$. We remark that the relation (2.5) with $r = d(1-1/q)$ and the Hölder inequality imply

$$\sum_{n\le|k|<2n} |\hat{f}(k)| = o(1). \tag{2.10}$$

First, observe that the cubic Fourier sums $S_n(f)$ uniformly converge to $f$ as $n \to \infty$. Indeed, let us consider the de la Vallée Poussin sums

$$V_n(f) = \sum_{|k|\le 2n} \prod_{j=1}^{d} \min\left(1, \frac{2n-|k_j|}{n}\right) \hat{f}(k) e^{i(k,x)}.$$

It is known (see [B]) that for any $f \in C(\mathbb{T}^d)$

$$\|V_n(f) - f\|_\infty = o(1), \quad (n \to \infty). \tag{2.11}$$

Further,

$$\|S_n(f) - V_n(f)\|_\infty \le \sum_{k\in\mathbb{Z}^d,\ n<|k|\le 2n} |\hat{f}(k)|,$$

and, by (2.10),

$$\|S_n(f) - V_n(f)\|_\infty = o(1), \quad (n \to \infty). \tag{2.12}$$

The relations (2.11) and (2.12) imply

$$\|S_n(f) - f\|_\infty = o(1) \quad (n \to \infty). \tag{2.13}$$

Thus, we obtain the uniform convergence of $S_n(f)$ to $f$.

The rest of the proof is similar to the above case $2 < p < \infty$ with the only difference that instead of the Hausdorff-Young theorem we use the inequality

$$\|f\|_\infty \le \sum_k |\hat{f}(k)|.$$

Theorem 1 is proved.

Let us discuss now a possibility of improving the assumption $f \in L_p(\mathbb{T}^d) \cap \mathcal{F}o_q^r$, $r = d(1/p' - 1/q)$, in Theorem 1.

**Proposition 2.1.** *For each $2 < p \le \infty$ there exists $f \in L_p(\mathbb{T}^d)$ such that*

$$|\hat{f}(k)| = O(|k|^{-d(1-1/p)}), \tag{2.14}$$

*(and, therefore, $f \in \mathcal{DF}^r_q$, $r = d(1/p' - 1/q)$) and the sequence $\{G_m(f)\}$ diverges in the $L_p$.*

*Proof.* We will use a construction from [T1]. We use the Rudin-Shapiro polynomials:

$$\mathcal{R}_N(x) = \sum_{|k| \le N} \epsilon_k e^{ikx}, \quad \epsilon_k = \pm 1, \quad x \in \mathbb{T}, \tag{2.16}$$

which satisfy the estimate

$$\|\mathcal{R}_N\|_\infty \le CN^{1/2}, \tag{2.17}$$

for an absolute constant $C$. Denote for $s = \pm 1$

$$\Lambda_{\pm 1} := \{k : \hat{\mathcal{R}}_m(k) = \pm 1\}.$$

The estimate (2.17) implies

$$||\Lambda_1| - |\Lambda_{-1}|| = |\mathcal{R}_m(0)| \le Cm^{1/2}. \tag{2.18}$$

Let $s = \pm 1$ be such that $|\Lambda_s| > |\Lambda_{-s}|$. Then take a small positive parameter $\delta$ and consider the function

$$f_{m,\delta} := \mathcal{R}_m + s\delta\mathcal{D}_m, \tag{2.19}$$

where

$$\mathcal{D}_m(x) := \sum_{|k| \le m} e^{ikx}, \quad x \in \mathbb{T},$$

is the Dirichlet kernel. Then since $|\hat{f}_{m,\delta}(k)| = 1 + \delta$ for $k \in \Lambda_s$ and $|\hat{f}_{m,\delta}(k)| = 1 - \delta$ for $k \in \Lambda_{-s}$ and $|\Lambda_s| \ge m$ the frequencies of $G_m(f_{m,\delta})$ will be in $\Lambda_s$ and

$$\|G_m(f_{m,\delta})\|_\infty \ge |G_m(f_{m,\delta})(0)| = (1 + \delta)m. \tag{2.20}$$

Next,

$$\|f_{m,\delta}\|_p \le \|\mathcal{R}_m\|_p + \delta\|\mathcal{D}_m\|_p \le \|\mathcal{R}_m\|_\infty + \delta\|\mathcal{D}_m\|_2^{2/p}\|\mathcal{D}_m\|_\infty^{1-2/p} \le \tag{2.21}$$

$$Cm^{1/2} + \delta(2m+1)^{1-1/p} \le C_1 m^{1/2}$$

for $\delta \le m^{1/p-1/2}$. By the Nikol'skii inequality for trigonometric polynomials the relation (2.20) implies

$$\|G_m(f_{m,\delta})\|_p \ge C_2 m^{-1/p}\|G_m(f_{m,\delta})\|_\infty \ge C_2 m^{1-1/p}. \tag{2.22}$$

Define now

$$f^d_{m,\delta}(x) := \prod_{j=1}^d f_{m,\delta}(x_j)e^{i(4m)x_j}$$

and

$$f := \sum_{l=1}^\infty 2^{-d(1-1/p)l} f^d_{2^l,\delta_l}(x), \quad 0 < \delta_l < 2^{-dl-3}.$$

The relation (2.14) is obviously satisfied. Moreover, (2.21) implies that

$$\|f - V_{2^n}(f)\|_\infty = O(2^{-d(1/2-1/p)n}). \tag{2.23}$$

However, (2.22) implies that $\{G_m(f)\}$ diverges in $L_p$.

Let us make some more comments. For a given set $\Lambda$ denote

$$E_\Lambda(f)_p := \inf_{c_k, k\in\Lambda} \|f - \sum_{k\in\Lambda} c_k e^{i(k,x)}\|_p.$$

**Remark 2.1.** *Theorem 1 implies that if $f \in L_p$, $2 < p \le \infty$, and*

$$E_{Q(n)}(f)_2 = o(n^{-(1/2-1/p)}) \tag{2.24}$$

*then $G^t_m(f) \to f$ in $L_p$.*

Indeed, (2.24) is equivalent to $f \in \mathcal{F}o^r_2$ with $r = d(1/2 - 1/p)$.

**Remark 2.2.** *The proof of Proposition 2.1 (see (2.23)) implies that there is $f \in L_p(\mathbb{T}^d)$ such that*

$$E_{Q(n)}(f)_\infty = O(n^{1/p-1/2})$$

*and $\{G_m(f)\}$ diverges in $L_p$, $2 < p \le \infty$.*

**Remark 2.3.** *There exists a continuous function $f$, satisfying (2.10), such that $\{G_m(f)\}$ diverges in the uniform norm.*

*Proof.* We construct an example in the univariate case. Define

$$f := \sum_{k\ge 2} b_k$$

with

$$b_k := s_k^{-1/2} \sum_{l=1}^{s_k} 2^{-s_k} f_{2^{s_k},\delta_{s_k}} e^{i4^{s_k+l}x}$$

where $\{s_k\}$ is an increasing sequence such that all frequences of $b_{k+1}$ lie to the right of frequences of $b_k$. Then by (2.21) we get

$$\|b_k\|_\infty \le C_1 s_k^{1/2} 2^{-s_k/2}$$

and, therefore, $f \in C(\mathbb{T})$. The relation (2.10) is also satisfied. It is clear that

$$\max_m \|G_m(b_k)\|_\infty \ge s_k^{1/2}.$$

This implies the divergence of $\{G_m(f)\}$.

We note that Remark 2.1 can also be obtained from some general inequalities for $\|f - G_m(f)\|_p$. We now define the $m$-term best approximation, i.e. the quantity

$$\sigma_m(f)_p := \inf_{k^j \in \mathbb{Z}^d, c_j} \|f - \sum_{j=1}^m c_j e^{i(k^j,x)}\|_p.$$

It has been proved in [T1] that for any $f \in L_p(\mathbb{T}^d)$ one has

$$\|f - G_m(f)\|_p \le (1 + 3m^{h(p)})\sigma_m(f)_p, \quad 1 \le p \le \infty,$$

where $h(p) := |1/2 - 1/p|$. Similarly to the above inequality one can prove the following relation.

**Theorem 2.2.** *For each $f \in L_p(\mathbb{T}^d)$ and any $0 < t \le 1$ we have*

$$\|f - G_m^t(f)\|_p \le (1 + (2 + 1/t)m^{h(p)})\sigma_m(f)_p, \quad 1 \le p \le \infty,$$

*where $h(p) := |1/2 - 1/p|$.*

*Proof.* This proof repeats the proof of Theorem 2.1 from [T1] that corresponds to the case $t = 1$ with one minor change. Let

$$G_m^t(f) = \sum_{k \in \Lambda'(t)} \hat{f}(k) e^{i(k,x)}, \quad |\Lambda'(t)| = m, \quad \Lambda' := \Lambda'(1).$$

Then the change in the proof from [T1] ($t = 1$) to adjust it for $t < 1$ is the following. Instead of obvious relation (see (2.10) from [T1]): for any $\Lambda$, $|\Lambda| = m$ one has

$$\|S_{\Lambda \setminus \Lambda'}(f)\|_2 \le \|S_{\Lambda' \setminus \Lambda}(f)\|_2$$

we use the inequality ($\Lambda$ is any, $|\Lambda| = m$)

$$\|S_{\Lambda \setminus \Lambda'(t)}(f)\|_2 \le t^{-1} \|S_{\Lambda'(t) \setminus \Lambda}(f)\|_2 \tag{2.25}$$

which follows easily from the definition of $\Lambda'(t)$.

We will prove one more inequality.

**Proposition 2.2.** *Let $2 \le p \le \infty$. Then for any $f \in L_p(\mathbb{T}^d)$ and any $Q$, $|Q| \le m$, we have*

$$\|f - G_m^t(f)\|_p \le \|f - S_Q(f)\|_p + (3 + 1/t)(2m)^{h(p)} E_Q(f)_2.$$

*Proof.* Let as above

$$G_m^t(f) = \sum_{k \in \Lambda'(t)} \hat{f}(k) e^{i(k,x)}.$$

Then

$$\|f - G_m^t(f)\|_p \le \|f - S_Q(f)\|_p + \|S_Q(f) - S_{\Lambda'(t)}(f)\|_p \tag{2.26}$$

and by Lemma 2.2 from [T1]

$$\|S_Q(f) - S_{\Lambda'(t)}(f)\|_p \le (2m)^{h(p)} \|S_Q(f) - S_{\Lambda'(t)}(f)\|_2. \tag{2.27}$$

Next,

$$\|S_Q(f) - S_{\Lambda'(t)}(f)\|_2 \le \|f - S_Q(f)\|_2 + \|f - S_{\Lambda'(t)}(f)\|_2. \tag{2.28}$$

Using (2.25) with $\Lambda = \Lambda'$ we get

$$\|S_{\Lambda'(t)}(f) - S_{\Lambda'}(f)\|_2^2 = \|S_{\Lambda'(t)\setminus\Lambda'}(f)\|_2^2 + \|S_{\Lambda'\setminus\Lambda'(t)}(f)\|_2^2 \le$$
$$(1 + t^{-2})\|S_{\Lambda'(t)\setminus\Lambda'}(f)\|_2^2 \le (1 + t^{-2})\sigma_m(f)_2^2.$$

Therefore,

$$\|f - S_{\Lambda'(t)}(f)\|_2 \le \|f - S_{\Lambda'}(f)\|_2 + \|S_{\Lambda'(t)}(f) - S_{\Lambda'}(f)\|_2 \le$$
$$\le (2 + 1/t)\sigma_m(f)_2 \le (2 + 1/t)E_Q(f)_2. \tag{2.29}$$

Combining (2.26)–(2.29) we complete the proof of Proposition 2.2.

We study now the convergence of greedy approximations of univariate functions of bounded $\Phi$-variation. Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function and $\Phi(0) = 0$. The class $V_\Phi$ of functions of bounded $\Phi$-variation is defined as the set of functions $f$ defined on $\mathbb{T}$ such that

$$v_\phi(f) = \sup \sum_j \Phi(|f(b_j) - f(a_j)|) < \infty,$$

where supremum is taken over all possible finite systems of disjoint intervals $(a_j, b_j) \subset \mathbb{T}$. For $\Phi(u) = u$ the class $V_\Phi$ is the class of functions of bounded variation. Clearly, if $\Phi_1(u) \le C\Phi_2(u)$, then $V_{\Phi_2} \subset V_{\Phi_1}$.

The classical Dirichlet—Jordan test asserts that if $f \in C(\mathbb{T})$ is a function of bounded variation then the Fourier series of $f$ uniformly converges to $f$ (see [Z, p. 57]). The convergence of Fourier series for functions of bounded $\Phi$-variation was studied by many authors; see related references in [O] where it was shown that the uniform convergence of Fourier series on the class $C(\mathbb{T}) \cap V_\Phi$ is equivalent to the condition

$$\int_0^1 \log(1/\Phi(u))du < \infty.$$

We proceed to a proposition that shows that we need a stronger restriction that the above one on the function $\Phi$ for convergence of greedy approximations.

**Proposition 2.3.** *a) If $u^2 = o(\Phi(u))$, $(u \to 0)$, and $f \in C(\mathbb{T}) \cap V_\Phi$, then*

$$\|f - G_m^t(f)\|_\infty \to 0 \quad as \quad m \to \infty.$$

*b) For $\Phi(u) = u^2$ there exists a function $f \in C(\mathbb{T}) \cap V_\Phi$ such that its greedy approximants $\{G_m(f)\}$ diverge at the point $x = 0$.*

*Proof.* Let $1 \le p \le \infty$, $\delta > 0$, $\omega(f,\delta)_p$ be the modulus of continuity of $f$ in $L_p$:

$$\omega(f,\delta)_p = \sup_{0 \le h \le \delta} \|f(\cdot + h) - f(\cdot)\|_p.$$

Let us estimate the modulus of continuity $\omega(f,\delta)_2$ of $f$ from $C(\mathbb{T}) \cap V_\Phi$. Take $h > 0$ and $n = [2\pi/h] + 1$. We have

$$\begin{aligned}
\|f(\cdot + h) - f(\cdot)\|_2^2 &\le \int_0^{nh} |f(t+h) - f(t)|^2 dt = \sum_{j=1}^{n} \int_{(j-1)h}^{jh} |f(t+h) - f(t)|^2 dt \\
&= \int_0^h \left( \sum_{j=1}^n |f(t + jh) - f(t + (j-1)h)|^2 \right) dt \\
&= \int_0^h o\left( \sum_{j=1}^n \Phi(|f(t+jh) - f(t + (j-1)h)|) \right) dt \\
&= \int_0^h o(2v_\phi(f)) dt = o(h).
\end{aligned}$$

Thus, $\omega(f,\delta)_2 = o(\sqrt{\delta})$ as $\delta \to 0$, and, by Jackson's theorem [A, p. 200],

$$E_n(f)_2 = o(n^{-1/2}).$$

This means that $f$ satisfies (2.24) with $p = \infty$. By Remark 2.1 $\|f - G_m^t(f)\|_\infty \to 0$ as $m \to \infty$.

To prove b), we use an example from Remark 2.2 with $p = \infty$. We have $E_n(f)_\infty = O(n^{-1/2})$. By Bernstein's theorem [A, p. 206], this implies $\omega(f,\delta)_\infty \le C\sqrt{\delta}$ for some $C$. We show that $\Phi$-variation of $f$ is finite for $\Phi(u) = u^2$. Indeed, for any disjoint intervals $(a_j, b_j)$

$$\sum_j |f(b_j) - f(a_j)|^2 \le \sum_j C^2 |b_j - a_j| \le 2\pi C^2,$$

and $v_\phi(f) \le 2\pi C^2$. This completes the proof of Proposition 2.3.

In particular, Proposition 2.3 implies that weak greedy approximations converge for any absolutely continuous function $f \in C(\mathbb{T})$. The same is true for $f \in C(\mathbb{T}^2)$. We use the notion of absolute continuity of a function of several variables suggested by L. Zajichek and developed in [H]. Let $\gamma \in (0,1)$. We say that a function $f : \mathbb{T}^d \to \mathbb{C}$ is absolutely continuous if for each $\varepsilon > 0$ there is $\delta > 0$ such that for each disjoint family $\{B_j := B(t_j, r_j)\}$ of balls in $\mathbb{T}^d$ the inequality $\sum_j V(B_j) < \delta$ implies

$$\sum_j \left( \sup_{t', t'' \in B(t_j, \gamma r_j)} |f(t') - f(t'')| \right)^d < \varepsilon,$$

where $B(t,r) = \{t' : |t - t'| \le r\}$ and $V(B)$ is the $d$-dimensional volume of the ball $B$. It is proven in [H] that the definition does not depend on $\gamma$ and for $d = 1$ coincides with the classical definition.

**Proposition 2.4.** *a) If $f$ is absolutely continuous on $\mathbb{T}^2$, then $\|f - G^t_m(f)\|_\infty \to 0$ as $m \to \infty$.*
*b) For $d > 2$ there exists an absolutely continuous on $\mathbb{T}^d$ function $f$ such that its greedy approximants $\{G_m(f)\}$ diverge at the point $x = 0$.*

*Proof.* It is shown in [H] that for any absolutely continuous function $f \in C(\mathbb{T}^d)$ its gradient belongs to $L_d(\mathbb{T}^d)$. Therefore, in the case $d = 2$ this implies (see [N]) that

$$E_{Q(n)}(f)_2 = o(n^{-1/2}),$$

and by Remark 2.1 we have

$$\|f - G^t_m(f)\|_\infty \to 0 \quad \text{as} \quad m \to \infty.$$

Further, from the example in Remark 2.2 it is easy to see that for $d > 2$ there exists a continuously differentiable on $\mathbb{T}^d$ function $f$ whose greedy approximations diverge at the point $x = 0$. It follows from the definition that the class of absolutely continuous functions contains all continuously differentiable (and, moreover, all Lipshitzian) functions. This proves the proposition.

## 3. Conditions in terms of decreasing rearrangement of Fourier coefficients. Proof of Theorem 2

Let us begin with the proof of Theorem 2. We reformulate it here for the convenience.

**Theorem 2.** *Let $2 < p < \infty$ and let a decreasing sequence $\{A_n\}_{n=1}^\infty$ satisfy the condition:*

(3.1) $$A_n = o(n^{1/p-1}) \quad \textit{as} \quad n \to \infty.$$

*Then for any $f \in L_p(\mathbb{T}^d)$ with the property $a_n(f) \le A_n$, $n = 1, 2, \dots,$ we have*

(3.2) $$\lim_{m \to \infty} \|f - G^t_m(f, \mathcal{T})\|_p = 0.$$

*Proof.* By the M. Riesz theorem (see [KS,Chap.4,S.3])we have for any $f \in L_p(\mathbb{T}^d)$, $1 < p < \infty$, that

$$\|f - S_N(f)\|_p \to 0 \quad \text{as} \quad N \to \infty. \tag{3.3}$$

We will consider first the case $t = 1$. Let us estimate $\|S^d_m(f) - G_m(f)\|_p$. Denote $\Sigma_1 := S^d_m(f - G_m(f))$ and $\Sigma_2 := (Id - S^d_m)(G_m(f))$. Then we have

$$S^d_m(f) - G_m(f) = S^d_m(f) - S^d_m(G_m(f)) - (Id - S^d_m)(G_m(f)) = \Sigma_1 - \Sigma_2.$$

For the first sum we get by the Paley theorem (see [Z,Chap.12,S5])

$$\|\Sigma_1\|_p \le C(p,d)(\sum_{n=1}^{2m+1} a_m(f)^p n^{p-2})^{1/p} = O(a_m(f)m^{1-1/p}) = o(1). \tag{3.4}$$

We now proceed to the second sum $\Sigma_2$. We first prove one general inequality.

**Proposition 3.1.** *Let $2 \le p < \infty$ and $u \in L_p$, $\|u\|_p \ne 0$. Then for any $v \in L_p$ we have*

$$\|u\|_p \le \|u + v\|_p + (\|u\|_{2p-2}/\|u\|_p)^{p-1}\|v\|_2.$$

*Proof.* Denote

$$F := \|u\|_p^{1-p}\bar{u}|u|^{p-2}.$$

Then

$$\|F\|_{p'} = 1 \quad \text{and} \quad \langle F, u\rangle = \|u\|_p.$$

Therefore,

$$\|u\|_p = \langle F, u\rangle = \langle F, u + v\rangle - \langle F, v\rangle \le \|u + v\|_p + \|F\|_2\|v\|_2.$$

It remains to observe that

$$\|F\|_2 = (\|u\|_{2p-2}/\|u\|_p)^{p-1}.$$

**Lemma 3.1.** *Let $2 \le p < \infty$. For $f \in L_p(\mathbb{T}^d)$ assume that $a_n(f) = o(n^{1/p-1})$. Then*

$$\|(Id - S^d_m)(G_m(f))\|_p = o(1).$$

*Proof.* We use Proposition 3.1 with

$$u := (Id - S^d_m)(G_m(f)); \quad v := f - S^d_m(f) - u.$$

Then

$$\|v\|_2 \le \|f - G_m(f)\|_2 \le (\sum_{n>m} a_n(f)^2)^{1/2} = o(m^{1/p-1/2}). \tag{3.5}$$

By the Paley theorem

$$\|u\|^{p-1}_{2p-2} = O((\sum_{n=1}^{m} a_n(f)^{2p-2}n^{2p-4})^{1/2}) = o(m^{1/2-1/p}). \tag{3.6}$$

Combining (3.5) and (3.6) and taking into account that $\|u + v\|_p = \|f - S^d_m(f)\|_p = o(1)$ we get by Proposition 3.1 that $\|u\|_p = o(1)$. Lemma 3.1 is now proved.

The required estimate $\|\Sigma_2\|_p = o(1)$ follows from Lemma 3.1. This together with (3.4) complete the proof of Theorem 2 in the case $t = 1$. The general case $0 < t \le 1$ follows from the case $t = 1$ and Lemma 3.2 below.

**Lemma 3.2.** *Let $2 \le p < \infty$, $t \in (0,1]$, and $f \in L_p(\mathbb{T}^d)$ be such that $a_n(f) = o(n^{1/p-1})$. Then*

$$\|G_m(f) - G_m^t(f)\|_p \to 0 \quad as \quad m \to \infty.$$

*Proof.* Let

$$G_m(f) = S_\Lambda(f) \quad \text{and} \quad G_m^t(f) = S_{\Lambda(t)}(f).$$

Then

$$g_m := G_m(f) - G_m^t(f) = \sum_{k\in\Lambda\setminus\Lambda(t)} \hat{f}(k)e^{i(k,x)} - \sum_{k\in\Lambda(t)\setminus\Lambda} \hat{f}(k)e^{i(k,x)}.$$

It is clear that

$$|\hat{f}(k)| \le a_m(f), \quad k \in \Lambda(t) \setminus \Lambda.$$

The relation (2.8) implies

$$|\hat{f}(k)| \le t^{-1}a_m(f), \quad k \in \Lambda \setminus \Lambda(t).$$

Thus, for the Fourier coefficients of the function $g_m$ we have

$$|\hat{g}_m(k)| \le t^{-1}a_m(f).$$

Taking into account that $g_m$ has at most $2m$ terms we get from the Paley theorem that

$$\|g_m\|_p = O(a_m(f)m^{1-1/p}) = o(1).$$

This proves the lemma.

Let us note that by the Hausdorff-Young theorem the condition

$$\sum_{n=1}^{\infty} A_n^{p'} < \infty, \qquad 2 < p < \infty,$$

which is stronger than (3.1) implies that for any $f$ such that $a_n(f) \le A_n$ its Fourier series converges in $L_p$ unconditionally.

**Proposition 3.2.** *Let a decreasing sequence $\{A_n\}_{n=1}^\infty$ does not satisfy the condition (3.1) of Theorem 2, i.e.,*

$$\limsup_{n\to\infty} A_n n^{1-1/p} > 0.$$

*Then there is a continouos function $f \in C(\mathbb{T})$ with the property $a_n(f) \le A_n$, $n = 1, 2, \dots$, such that $\{G_m(f)\}$ diverges in the $L_p$-norm, $2 < p < \infty$.*

*Proof.* We will use functions constructed in the proof of Proposition 2.1. Let a number $c > 0$ and a sequence $\{n_k\}$ be such that

$$A_{n_k} \ge c n_k^{1/p-1}, \quad n_k \ge 4n_{k-1}, \quad n_1 \ge 4.$$

Define $m_k := [n_k/4]$ and

$$f := c\sum_{k=1}^{\infty} n_k^{1/p-1} f_{m_k,\delta_k} e^{in_k x}$$

where $f_{m,\delta}$ are defined by (2.19). Then $f$ is a continuous function, satisfying the property $a_n(f) \le A_n$. Divergence of $\{G_m(f)\}$ follows from (2.22).

## 4. Conditions in terms of decreasing rearrangement of Fourier coefficients. Proof of Theorems 3–5

We begin with the proof of Theorem 3. We reformulate it here for the convenience.

**Theorem 3.** *Let a decreasing sequence $\{A_n\}_{n=1}^{\infty}$ satisfy the condition ($\mathcal{A}_\infty$):*

$$\sum_{M<n\le e^M} A_n = o(1) \quad as \quad M\to\infty. \tag{4.1}$$

*Then for any $f\in C(\mathbb{T})$ with the property $a_n(f)\le A_n$, $n=1,2,\dots,$ we have*

$$\lim_{m\to\infty}\|f-S_{\Lambda_m}(f)\|_\infty = 0, \tag{4.2}$$

*where $\Lambda_m$ is an arbitrary subset of $\mathbb{Z}$ satisfying*

$$|\Lambda_m| = m, \tag{4.3}$$

$$\min_{k\in\Lambda_m}|\hat f(k)| \ge t \max_{k\notin\Lambda_m}|\hat f(k)|. \tag{4.4}$$

*Proof.* Denote as above

$$G_m(f) = \sum_{n=1}^{m}\hat f(k(n))e^{ik(n)x}.$$

Note that if $k\neq k(n)$ for $n\le m$ then $|\hat f(k)|\le a_m(f)$. Also, by (4.4), if $k\notin\Lambda_m$ then $|\hat f(k)|\le a_m(f)/t$. Therefore,

$$\|S_{\Lambda_m}(f)-G_m(f)\|_\infty \le m a_m(f) + m a_m(f)/t. \tag{4.5}$$

It is clear that (4.1) implies $A_n = o(n^{-1})$ and, therefore,

$$a_m(f)m = o(1). \tag{4.6}$$

Relations (4.5) and (4.6) give

$$\|S_{\Lambda_m}(f)-G_m(f)\|_\infty = o(1). \tag{4.7}$$

Let us estimate $\|V_m(f)-G_m(f)\|_\infty$, where $V_m(f)$ is the de la Vallée Poussin sum

$$V_m(f) = \sum_{|k|\le 2m}\min\left(1,\frac{2m-|k|}{m}\right)\hat f(k)e^{ikx}.$$

We have

$$V_m(f) - G_m(f) = \Sigma_1 - \Sigma_2,$$

where

$$\Sigma_1 = V_m(f - G_m(f)),$$

$$\Sigma_2 = (Id - V_m)(G_m(f)).$$

For the first sum we get

$$\|\Sigma_1\|_\infty \le \sum_{n=1}^{4m-1} a_m(f) \le 4m a_m(f).$$

Therefore, by (4.6), $\|\Sigma_1\|_\infty = o(1)$.

We proceed to the second sum $\Sigma_2$. Let us consider

$$f - V_m(f) - \Sigma_2 = \sum_{m<n\le e^{e^m}, k(n)>m} \lambda_n \hat{f}(k(n)) e^{ik(n)x} + g =: \Sigma_4 + g, \tag{4.8}$$

where $0 \le \lambda_n \le 1$. Using (2.11) and the assumption $\mathcal{A}_\infty$ we get from (4.8)

$$\|\Sigma_2 + g\|_\infty \le \|f - V_m(f)\|_\infty + \|\Sigma_4\|_\infty = o(1). \tag{4.9}$$

Next we have

$$\|g\|_2 \le (\sum_{n>e^{e^m}} a_n(f)^2)^{1/2} = o\left(e^{-e^m/2}\right). \tag{4.10}$$

We need the following lemma that we will prove a little later.

**Lemma 4.1.** *Let a function $f$, $\|f\|_\infty = 1$, have a form*

$$f = \sum_{k\in\Lambda} \hat{f}(k) e^{ikx}, \qquad |\Lambda| \le m.$$

*Then for any function $g$ such that $\|g\|_2 \le \frac{1}{4}(4\pi m)^{-m/2}$ we have*

$$\|f + g\|_\infty \ge 1/4.$$

This lemma and (4.9) imply that $\|\Sigma_2\|_\infty = o(1)$. Together with (4.7) this completes the proof of Theorem 3.

*Proof of Lemma 4.1.* Denote by $\|u\|$ the distance from a real number $u$ to the closest integer. Denote for a fixed $j \in \mathbb{N}$

$$F_j = \{x \in \mathbb{T} : \ \forall k \in \Lambda, \quad \|j(kx/(2\pi))\| < 1/(4\pi m)\},$$

$$F = F_1.$$

Well-known estimates for simultaneous diophantine approximation (see [C, p. 13]) give

$$\mathbb{T} = \bigcup_{j \le J} F_j, \quad J = (4\pi m)^m.$$

Note that $\mu F_j = \mu F$ for all $j$. Therefore,

$$1 \le \sum_{j \le J} \mu F_j \le J \mu F,$$

or,

$$\mu F \ge (4\pi m)^{-m}. \tag{4.11}$$

Let $|f(x_0)| = \|f\|_\infty = 1$, $E = \{x_0 + y : \ y \in F\}$. For $x = x_0 + y \in E$, $k \in \Lambda$ we have

$$\left| e^{ikx} - e^{ikx_0} \right| \le 2\pi \|ky/(2\pi)\| < 1/(2m).$$

Therefore,

$$|f(x) - f(x_0)| \le \sum_{k \in \Lambda} |\hat{f}(k)| \left| e^{ikx} - e^{ikx_0} \right| \le \sum_{k \in \Lambda} (1/2m) \le 1/2.$$

Thus, $|f(x)| \ge 1/2$ for $x \in E$.

Suppose that

$$\|f + g\|_\infty < 1/4. \tag{4.12}$$

Then $|g(x)| > 1/4$ for $x \in E$, and, by (4.11),

$$\|g\|_2^2 \ge \int_E |g(x)|^2 d\mu > \left(\frac{1}{4}\right)^2 (4\pi m)^{-m}.$$

This does not agree with the condition of the lemma. Hence, (4.12) is not true, and the proof is complete.

**Remark 4.1.** *Actually, in the proof of Lemma 4.1 we have shown the following. If*

$$f = \sum_{k \in \Lambda} \hat{f}(k) e^{ikx}, \qquad |\Lambda| \le m,$$

$G \subset \mathbb{T}$, $\mu G > 1 - (4\pi m)^{-m}$, *then*

$$\|f\|_\infty \le 2 \sup_{x \in G} |f(x)|.$$

*Recently the first author and Nazarov (not published) have proved that the last inequality holds under an assumption* $\mu G > 1 - c^m$ *for a small constant* $c$. *This can be used to weaken the assumption on* $\|g\|_2$ *in Lemma 4.1. However it does not affect Theorem 3.*

We proceed to the proof of Theorem 4 from Introduction. The core part of the proof of Theorem 4 is the following lemma.

**Lemma 4.2.** *Fix $\Delta > 0$, $\delta > 0$. Let positive integers $m \to \infty$ and $M \to \infty$ be such that*

$$\log M = o(m). \tag{4.13}$$

*Let $m_1 = m$, $m_3 = m + M$, $m_1 < m_2 < m_3$. Let a decreasing sequence $\{A_n\}_{n=1}^{\infty}$ satisfy the conditions*

$$A_n \le \Delta/n, \tag{4.14}$$

$$\sum_{n=m_1+1}^{m_2} A_n = \sum_{n=m_2+1}^{m_3} A_n = 1, \tag{4.15}$$

$$A_{2m} > \delta A_m. \tag{4.16}$$

*Then for sufficiently large $m$ there exists a trigonometric polynomial*

$$T(x) = T_m(x) = \sum_{k=1}^{M} \hat{T}(k) e^{i(k,x)}$$

*such that*

$$a_k(T) \le A_{m+k} \quad (1 \le k \le M), \tag{4.17}$$

$$\|T\|_\infty \to 0 \quad (m \to \infty), \tag{4.18}$$

$$\max_m |G_m(T, \mathcal{T})(0)| > 0.01. \tag{4.19}$$

*Proof.* Take independent random variables $\eta_k$ $(1 \le k \le M)$ so that each $\eta_k$ is equal to any $n$, $m_1 < n \le m_3$ with probability $1/(10M)$, and is equal to $m_1$ with probability 0.9. A polynomial $T$ is defined as

$$T(x) = \sum_{k=1}^{M} \sigma_{\eta_k} A_{\eta_k} e^{i(k,x)},$$

where $\sigma(m_1) = 0$, $\sigma_n = 1$ for $m_1 < n \le m_2$, $\sigma_n = -1$ for $m_2 < n \le m_3$. We prove that a polynomial $T$ satisfies the conditions (4.17)—(4.19) with a large probability. Probability, expectation and variance will be denoted by $P$, $E$, and $V$, respectively. We will estimate the probailities of the following events:

$$E_1 : \quad \exists l \ge 1 : |\{k : m_1 < \eta_k \le m_1 + l\}| > l,$$

$$E_2 : \quad \|T\|_\infty > 3(A_m \log(2\pi M^2))^{1/2},$$

$$E_3 : \qquad \sum_{k: m_1 < \eta_k \le m_2} A_{\eta_k} \le 0.05.$$

Note that nonfulfillment of the events $E_1$, $E_2$, $E_3$ imply (4.17), (4.18), (4.19), respectively. In the case of $E_2$ and (4.18) we use (4.13) and (4.14) to prove that $A_m \log(2\pi M^2) = o(1)$.

Consider the event

$$E_{1,l} : |\{k : m_1 < \eta_k \le m_1 + l\}| > l.$$

We have

$$P(E_1) \le \sum_l P(E_{1,l}). \tag{4.20}$$

Further,

$$\begin{aligned} P(E_{1,l}) &= \sum_{j=l+1}^{M} \binom{M}{j} \left(\frac{l}{10M}\right)^j \left(1 - \frac{l}{10M}\right)^{M-j} \\ &\le \sum_{j=l+1}^{M} \binom{M}{j} \left(\frac{l}{10M}\right)^j. \end{aligned} \tag{4.21}$$

For any $j > l$ we have

$$\left(\frac{l}{10M}\right)^j < 10^{-l} \left(\frac{l}{M}\right)^j,$$

$$1 \le e^l \left(1 - \frac{l}{M}\right)^{M-l} \le e^l \left(1 - \frac{l}{M}\right)^{M-j}.$$

Therefore,

$$\sum_{j=l+1}^{M} \binom{M}{j} \left(\frac{l}{10M}\right)^j \le (e/10)^l \sum_{j=l+1}^{M} \binom{M}{j} \left(\frac{l}{M}\right)^j \left(1 - \frac{l}{M}\right)^{M-j} \le (e/10)^l.$$

By (4.20) and (4.21) we get

$$P(E_1) \le \sum_l (e/10)^l < 1/2. \tag{4.22}$$

To estimate $P(E_2)$, we use the following theorem [Ka, pp. 68, 79].

**Theorem A.** *Let $E$ be a measurable space with measure $\mu$, and $\mu(E) < \infty$. Let $B$ be a linear space of measurable bounded functions on $E$, closed under complex conjugation, and suppose that there exists $\rho > 0$ with the following property: if $f \in B$ and $f$ is real, then there exists a measurable set $I = I(f) \subset E$ such that $\mu(I) > \mu(E)/\rho$ and $|f(t)| \geq \frac{1}{2}\|f\|_\infty$ for $t \in I$. Let us consider a random finite sum*

$$P = \sum \xi_k f_k,$$

*where*

$$E(\xi_k) = 0, \quad E(\xi_k^2) = b_k^2, \quad |\xi_k| \leq 1.$$

*Moreover we suppose that $\|f_k\|_\infty = 1$ and $r = \sum b_k^2 > \log \rho$. Then*

$$P\left(\|P\|_\infty \geq 6(r \log \rho)^{1/2}\right) \leq \frac{4}{\rho}.$$

We use Theorem A for $E = \mathbb{T}$, $B = \{\sum_{k=1}^{M} c_k e^{i(k,x)}\}$, $f_k = e^{i(k,x)}$, $\xi_k = \sigma_{\eta_k} A_{\eta_k}/A_m$. Note that

$$P(x) = T(x)/A_m. \tag{4.23}$$

One can guarantee the existence of the required set $I(f)$ by taking

$$\rho = 2\pi M^2 \tag{4.24}$$

([Ka, p. 49]). Further, for $k = 1, \dots, M$ we have $E\xi_k = 0$, and, by (4.16),

$$b_k^2 = E\xi_k^2 = \frac{1}{10MA_m^2} \sum_{n=m_1+1}^{m_3} A_n^2 \geq \frac{m\delta^2}{10M}.$$

Therefore, $r > \frac{m\delta^2}{10}$, and, by (4.13) and (4.24), for sufficiently large $m$ the condition $r > \log \rho$ holds. On the other hand,

$$\sum_{n=m_1+1}^{m_3} A_n^2 \leq A_m \sum_{n=m_1+1}^{m_3} A_n = 2A_m, \tag{4.25}$$

$b_k^2 \leq \frac{1}{5MA_m}$, and $r \leq \frac{1}{5A_m}$. Thus, by (4.23),

$$P\left(\|P\|_\infty \geq 6(r \log \rho)^{1/2}\right) \geq P\left(\|P\|_\infty \geq 3(\log(2\pi M^2)/A_m)^{1/2}\right)$$
$$= P\left(\|T\|_\infty \geq 3(A_m \log(2\pi M^2))^{1/2}\right),$$

and Theorem A gives

$$P(E_2) \le \frac{4}{\rho} \le M^{-2}. \tag{4.26}$$

To estimate $P(E_3)$, we define the random variables $\nu_1, \dots, \nu_M$ as $\nu_k = A_{\eta_k}$ for $m_1 < \eta_k \le m_2$ and $\nu_k = 0$ otherwise. The event $E_3$ can be rewritten as

$$E_3 : \quad \sum_{k=1}^{M} \nu_k \le 0.05.$$

We have

$$E(\nu_k) = \frac{1}{10M}$$

and, by (4.25),

$$V(\nu_k) \le E(\nu_k^2) \le \frac{A_m}{5M}.$$

Hence,

$$E\left(\sum_{k=1}^{M} \nu_k\right) = 0.1,$$

$$V\left(\sum_{k=1}^{M} \nu_k\right) \le \frac{A_m}{5},$$

and, by Chebyshev's inequality,

$$P(E_3) \le \frac{V\left(\sum_{k=1}^{M} \nu_k\right)}{\left(E\left(\sum_{k=1}^{M} \nu_k\right) - 0.05\right)^2} \le 80A_m. \tag{4.27}$$

So, by (4.22), (4.26), and (4.27), $P(E_1) + P(E_2) + P(E_3) < 1$, and there exists a choice of a polynomial $T$ for which neither of the events $E_1, E_2, E_3$ holds. This completes the proof of Lemma 4.2.

**Theorem 4.** *Assume that a decreasing sequence $\{A_n\}_{n=1}^{\infty}$ does not satisfy the condition $(\mathcal{A}_\infty)$. Then there exists a function $f \in C(\mathbb{T})$ with the property $a_n(f) \le A_n$, $n = 1, 2, \dots$, and such that we have*

$$\limsup_{m \to \infty} \|f - G_m(f, \mathcal{T})\|_\infty > 0.$$

*Proof.* Without loss of generality we may suppose that

$$\limsup_{u \to \infty} \sum_{u < n \le e^u} A_n > 8 \tag{4.28}$$

where $u \in \mathbb{R}$. Also, we may assume that for sufficiently large $n$

$$A_n \le 10/n. \tag{4.29}$$

Indeed, if (4.29) fails for infinitely many $n$'s, we replace all $A_n$ by $A'_n = \min(A_n, 10/n)$. If for a large $m$ we have $A_m > 10/m$, then

$$\sum_{\log m < n \le m} A'_n \ge \sum_{\log m < n \le m} 10/m > 9,$$

and (4.28) holds for $A'_n$. Now, observe that $F(F(u)) > e^u$ for sufficienly large $u$ and $F(u) = e^{\sqrt{u}}$. Therefore,

$$\sum_{u < n \le e^u} A_n \le \sum_{u < n \le F(u)} A_n + \sum_{F(u) < n \le F(F(u))} A_n,$$

and (4.28) implies

$$\limsup_{u \to \infty} \sum_{u < n \le e^{\sqrt{u}}} A_n > 4. \tag{4.30}$$

We will prove now that there exists an arbitrary large integer $m$ such that

$$\sum_{m < n \le e^{3\sqrt{m}}} A_n > 3 \tag{4.31}$$

and

$$A_{2m} \ge A_m/100. \tag{4.32}$$

Indeed, by (4.30), we can take a large $u$ with

$$\sum_{u < n \le e^{\sqrt{u}}} A_n > 4.$$

Let $m_0 = [u]$. If $A_{m_0} \ge 1/(2m_0)$, then the number $m = [m_0/2]$ satisfies (4.31) and (4.32) (we use (4.29) with $n = m$). If $A_{m_0} < 1/(2m_0)$, we define the sequence $m_j = 2^j m_0$. We take $m$ as the minimal $m = m_j$ satisfying (4.32). To show the existence of such an $m$ and to prove (4.31), we note that

$$\sum_{m_0 < n \le m_j} A_n < 1$$

whenever $A_{m_1} < A_{m_0}/100, \ldots, A_{m_j} < A_{m_{j-1}}/100$. Hence, the number $m$ does exist and, moreover,

$$\sum_{m < n \le e^u} A_n > 3,$$

which clearly implies (4.31).

We take now any large $m = m_1$ satisfying (4.31) and (4.32) and define

$$m_2 = \min\{m' : \sum_{m_1 < n \le m'} A_n \ge 1\},$$

$$m_3 = \min\{m' : \sum_{m_2 < n \le m'} A_n \ge \sum_{m_1 < n \le m_2} A_n\}.$$

We have

$$\sum_{m_1 < n \le m_3} A_n < 2 + 2a_{m_2} + a_{m_3} < 3.$$

This inequality combined with (4.31) shows that $m_3 < e^{3\sqrt{m}}$. We apply now Lemma 4.2 to the sequence $\{A'_n\}$, where

$$A'_n = \left( \sum_{m_1 < k \le m_2} A_k \right)^{-1} A_n \quad (n \le m_2),$$

$$A'_n = \left( \sum_{m_2 < k \le m_3} A_k \right)^{-1} A_n \quad (n > m_2).$$

We get a polynomial $T = T_m$ satisfying (4.17)—(4.19). Setting $f = \sum_m T_m$ where the sum is taken over a sparse sequence of $m$'s we complete the proof of Theorem 4.

**Theorem 5.** *Assume that a decreasing sequence $\{A_n\}_{n=1}^{\infty}$ is not summable. Then there exists a continuous function with the property $a_n(f) \le A_n$ and such that its partial Fouries sums diverge at some point.*

Theorem 5 is a simple corollary of the following lemma.

**Lemma 4.3.** *Let a decreasing sequence $\{A_n\}_{n=1}^{\infty}$ be not summable. Then for any $l \in \mathbb{N}$ and $m_0 \in \mathbb{N}$ there exist a trigonometric polynomial $T(x) = T_l(x)$ and numbers $m \ge m_0$, $N \in \mathbb{N}$ such that*

$$a_k(T) \le A_{m+k} \quad (k \ge 1), \tag{4.33}$$

$$\|T\|_\infty \to 0 \quad (l \to \infty), \tag{4.34}$$

$$|S_N(T, 0)| \to \infty \quad (l \to \infty). \tag{4.35}$$

*Proof of Lemma 4.3.* By the conditions on $\{A_n\}$ we have for any $l \in \mathbb{N}$

$$\sum_n A_{2ln} = \infty.$$

Therefore, for any $l > 1$ we can find $m_1 > m_0$ and $m_2 > m_1$ such that

$$(\log l)^{-1/2} \le \sum_{m_1 < n \le m_2} A_{2ln} \le 2(\log l)^{-1/2} \tag{4.36}$$

We associate with any $n$, $m_1 < n \le m_2$, a trigonometric polynomial

$$T_n(x) = A_{2ln} e^{ik_n x} \sum_{j=1}^{l} \frac{\sin(Kjx)}{j},$$

where the numbers $K$ and $k_n$ and a positive integer $N$ satisfy the conditions

$$k_n = N - n, \quad K > m_2, \quad N > lK.$$

We define

$$T = \sum_{m_1 < n \le m_2} T_n.$$

Let us prove prove (4.33) with $m = 2m_1$. We observe that by the choice of the numbers $k_n$ and $K$ the spectra of polynomials $T_n$ are disjoint, that is for any $j$ there exists at most one $n$ such that $\hat{T}_n(j) \neq 0$. We have

$$T_n(x) = \sum_{1 \le |j| \le l} \hat{T}_n(k_n + Kj) e^{i(k_n + Kj)x}, \quad |\hat{T}_n(k_n + Kj)| = A_{2ln}/(2|j|).$$

Therefore, we can write the following inequalities

$$|\hat{T}_n(k_n + Kj)| \le A_{2ln} \le A_{2ln-j} \quad (1 \le j \le l),$$

$$|\hat{T}_n(k_n - Kj)| \le A_{2ln} \le A_{2ln-n-j} \quad (1 \le j \le l)$$

and note that for $n > m_1$, $1 \le j \le l$, the numbers $2ln - j$, $2ln - n - j$ are all greater than $2m_1$ and pairwise distinct. This proves (4.33) with $m = 2m_1$.

We will check (4.34) and (4.35) now. Using the well known estimate

$$\left\| \sum_{j=1}^{l} \frac{\sin(ju)}{j} \right\|_{\infty} \le C$$

[Z, p. 61], we get

$$\|T\|_\infty \le C \sum_{m_1 < n \le m_2} A_{2ln},$$

and, by (4.36),

$$\|T\|_\infty \le 2C(\log l)^{-1/2}.$$

Let us estimate $S_N(T,0)$. We have

$$S_N(T_n,0) = \frac{i}{2} A_{2ln} \sum_{j=1}^{l} \frac{1}{j}.$$

Hence,

$$|S_N(T,0)| = \frac{1}{2} \sum_{m_1<n\le m_2} A_{2ln} \sum_{j=1}^{l} \frac{1}{j},$$

and (4.35) follows from (4.36). The proof is complete.

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