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Corner singularities and boundary layers in a simple convection-diffusion problem*

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Abstract

A singularly perturbed convection-diffusion problem posed on the unit square is considered. Its solution may have exponential and parabolic boundary layers, and corner singularities may also be present. Pointwise bounds on the solution and its derivatives are derived. The dependence of these bounds on the small diffusion coefficient, on the regularity of the data, and on the compatibility of the data at the corners of the domain are all made explicit. The bounds are derived by decomposing the solution into a sum of solutions of elliptic boundary-value problems posed on half-planes, then analyzing these simpler problems.

1. Introduction

This paper treats the following singularly perturbed convection-diffusion problem in the unit square $Q = (0,1) \times (0,1)$:

$$\begin{aligned} Lu := -\varepsilon\Delta u + pu_x + qu &= f \text{ in } Q, \\ u(x,0) = g_s(x),\ u(x,1) &= g_n(x) \text{ for } 0 < x < 1, \\ u(0,y) = g_w(y),\ u(1,y) &= g_e(y) \text{ for } 0 < y < 1. \end{aligned} \tag{1.1}$$

The coefficients p and q are positive constants while the parameter ε lies in $(0,1]$. The functions f, g_w, g_e, g_s, g_n are assumed to satisfy, for some non-negative integer ℓ and $\alpha \in (0,1)$,

$$f \in C^{2\ell,\alpha}(\bar{Q}),\ \ g_w, g_e, g_s, g_n \in C^{2\ell,\alpha}([0,1]). \tag{1.2}$$

A specified amount of compatibility between the boundary data and the solution is assumed at the 4 corners of Q. In particular, it may happen that $g_s(0) \neq g_w(0)$, etc. Consequently certain corner singularities form part of the solution to (1.1), and these will interact with the boundary layers induced by the convective nature of the problem.

The purpose of the paper is to give pointwise bounds on $\bar{Q}$ for the derivatives of the solution to (1.1) and to determine explicitly how these bounds depend on the parameter ε. As one would expect, the bounds at each point (x,y) also depend explicitly on the distance from (x,y) to the 4 sides and 4 corners of Q. Thus the bounds describe the effects of the corner singularities at the 4 vertices, the boundary layer at $x = 1$, and the characteristic boundary layers at $y = 0$ and $y = 1$.

One reason for this study is to understand the structures in the solution induced by the interaction between the corner singularites and the boundary layers. These structures are not revealed by an asymptotic expansion of the solution (as in, e.g., [4]), but they are revealed through a study of derivatives. The final result, Theorem 5.1, shows how, at an incoming corner, the corner singularity is propagated along the axis by a characteristic boundary layer, and how the corner singularity behaves near the intersection of the

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characteristic and outgoing boundary layers. These results would seem to illustrate, in the simplest case, what might take place in a fluid flow near entrant and exit corners.

A second reason for the study is that it is useful in numerical analysis. To analyze the discretization error of any numerical method for (1.1), derivative bounds are crucial. The bounds on derivatives in this paper can be used in the analysis of both finite element and finite difference methods. Furthermore, they suggest how to design an efficient mesh for the numerical solution of the problem. (This will be dealt with in a subsequent paper.) The bounds suggest the stretched mesh refinement that should be used in the several layer regions, and they suggest the mesh refinement strategy that should be used at both the "incoming corners" $(0,0)$ and $(0,1)$, and the "outgoing corners" $(1,0)$ and $(1,1)$. It seems plausible that similar mesh refinement strategies could be of use in more complicated problems with many layers and corner singularities.

Several authors have previously obtained bounds for derivatives of solutions of singularly perturbed problems in regions with corners. In the case $p = 0$, Han and Kellogg [3] gives pointwise bounds for the derivatives of the solution. The present paper may be considered an extension of [3] to the convective case. Linß and Stynes [7] consider a convection-diffusion problem in a square, with a non-horizontal convective direction. It is assumed that the data is compatible at the corners, so there is no internal parabolic layer and corner singularities are excluded. Bounds for the derivatives are derived. In Kellogg [5] the problem (1.1) is considered in an outgoing sector, which excludes boundary layers and so is an easier case. The analysis of Roos [9] is for the problem (1.1) in the case of compatible boundary conditions and outlines how pointwise bounds on derivatives might be obtained, but some of the arguments are unclear and certain critical details seem to be overlooked. Shih and Kellogg [10] give a detailed discussion of an asymptotic expansion for (1.1), with limited information on derivatives. A somewhat different and more revealing expansion is used in the present paper. Finally, Shishkin [11, Chapter IV] obtains pointwise bounds for derivatives of solutions of problems like (1.1) with variable coefficients, but compatibility conditions are assumed at the corners of the square.

Our methods use decompositions of the problem into various simpler elliptic problems. Thus, asympotic expansions, with their related ordinary and parabolic differential equations, are not used. Eqn. (5.11) expresses the solution as a sum of solutions to half-plane and quarter-plane problems plus a remainder. The remainder satisfies a problem of the form (1.1) with data that is compatible to all orders at the vertices and is exponentially small. Thus derivatives of (5.11) give a representation of derivatives of the solution u with exponentially small remainder.

The plan of the paper is as follows. Section 2 discusses the problem (1.1) in the positive quadrant, and with $f = 0$. This quarter-plane problem contains the essential difficulties of (1.1), and its solution is decomposed into a sum of three half-plane problems. In Section 3 two of these half-plane problems are analyzed using maximum principle arguments. The third half-plane problem is more difficult; its solution contains both corner singularities and a parabolic boundary layer. It is analyzed in Section 4, using a Green's function representation of the solution. Section 5 returns to (1.1), whose solution is decomposed into a sum of quarter-plane problems and half-plane problems. Each term in the sum is analyzed using the preceding results. The final result is presented in Theorem 5.2.

We shall use the Hölder space $C^{m,\alpha}(Q)$ where $0 < \alpha < 1$, the Sobolev space $H^m(Q)$ with norm $\|\cdot\|_{H^m(Q)}$, and for various sets S the Sobolev space $W^{m,\infty}(S)$ with norm $\|\cdot\|_{m,\infty,S}$. If $m = 0$ we write $\|\cdot\|_{\infty,S}$.

2. The quarter-plane problem

Let us denote the first quadrant by $\mathbb{Q} = (0,\infty) \times (0,\infty)$. In this section we are concerned with the quarter-plane problem

$$\begin{aligned} -\varepsilon\Delta u + pu_x + qu &= 0 \text{ in } \mathbb{Q}, \\ u(x,0) &= g(x) \text{ for } x > 0, \\ u(0,y) &= h(y) \text{ for } y > 0. \end{aligned} \tag{2.1}$$

We suppose that $g, h \in C^{2\ell,\alpha}(R^+)$ for some integer $\ell \geq 0$ and $\alpha \in (0,1)$, and satisfy for some constants $\bar{G}_\ell$ and $\bar{H}_\ell$ the inequalities

$$|g^{(k)}(x)| \leq \bar{G}_\ell, \ |h^{(k)}(y)| \leq \bar{H}_\ell \varepsilon^{-k/2} e^{-cy/\sqrt{\varepsilon}}, \text{ for } k = 0, \cdots, 2\ell. \tag{2.2}$$

We also assume that the data g, h satisfies the first $\nu + 1$ compatibility conditions for the problem (2.1). The zeroth compatibility condition is the continuity of the data at the origin: $g(0) = h(0)$. For $\nu > 0$ the compatibility conditions express compatibility of the differential equation and the boundary data at the origin. In the case that $g(x) \equiv 0$, the compatibility conditions are

$$h^{(2k)}(0) = 0 \text{ for } k = 0, \cdots, \nu. \tag{2.3}$$

From the theory of corner singularities, if the data satisfies the first $\nu + 1$ compatibility conditions with $\nu \leq \ell - 1$, then the solution u lies in $C^{2\nu+1,\alpha}(\bar{\mathbb{Q}})$, while $u \in C^{2\ell,\alpha}(\bar{\mathbb{Q}})$ if $\nu = \ell$. The value $\nu = -1$ is used to indicate that no compatibility condition is assumed. For the theory of corner singularities, see [2], and [3] in the case of a 90° angle.

In this section we give a decomposition of the solution that will enable us, in Section 4, to bound the derivatives in $\mathbb{Q}$. For the decomposition we start by extending g to a smooth function g_1 on $\mathbb{R}$, which vanishes for $x \leq -1$. By choosing C appropriately, we can also assume that $|g_1^{(i)}(x)| \leq Ce^{-a_1 x}$ for all $x \geq -1$ and $i = 0, 1, \ldots, 2\ell$, where a_1 is some positive constant that satisfies $a_1 \geq q/(2p)$. Let u_1 satisfy the "grazing" half-plane problem

$$\begin{aligned} -\varepsilon\Delta u_1 + pu_{1,x} + qu_1 &= 0 \text{ for } y > 0, \\ u_1(x,0) &= g_1(x) \text{ for } x \in \mathbb{R}. \end{aligned} \tag{2.4}$$

It will be shown in Theorem 3.10 that u_1 satisfies

$$|D_x^m D_y^n u_1(x,y)| \leq C\varepsilon^{-m-n/2} e^{-qx/(2p)} e^{-\sqrt{q}y/(2\sqrt{\varepsilon})} \bar{G}_{2\ell} \text{ for } m + n \leq 2\ell. \tag{2.5}$$

Let $h_1(y) = u_1(0,y)$ and let $u_2 = u - u_1$. Then u_2 satisfies the quarter-plane problem

$$\begin{aligned} -\varepsilon\Delta u_2 + pu_{2,x} + qu_2 &= 0 \text{ for } y > 0, \\ u_2(x,0) &= 0 \text{ for } x > 0, \\ u_2(0,y) &= h_2(y) := h(y) - h_1(y) \text{ for } y > 0. \end{aligned} \tag{2.6}$$

Using (2.2) and (2.5) with $m = 0$, it is seen that h_2 satisfies

$$|h_2^{(k)}(y)| \leq C\varepsilon^{-k/2} e^{-\sqrt{q}y/(2\sqrt{\varepsilon})} (\bar{G}_{2\ell} + \bar{H}_{2\ell}) \text{ for } k = 0 \cdots, 2\ell. \tag{2.7}$$

Since u_1 is a smooth function, u_2 has the same smoothness as u. Therefore the data of the problem (2.6) has the same compatibility at the origin as the data of the problem (2.1). From (2.3) the compatibility conditions for the problem (2.6) are $h_2^{(2k)}(0) = 0$ for $k = 0, \cdots, \nu$.

Let u_3 be the odd extension of u_2 to $y < 0$ and let h_3 be the odd extension of h_2 to $y < 0$. Then u_3 solves the incoming half-plane problem

$$\begin{aligned} -\varepsilon\Delta u_3 + pu_{3,x} + qu_3 &= 0 \text{ for } x > 0, \\ u_3(0,y) &= h_3(y) \text{ for } y \in \mathbb{R}. \end{aligned} \tag{2.8}$$

However h_3 or its derivatives may not be continuous on the y-axis. In fact, we see that if the first $\nu+1$ compatibility conditions of (2.1) are satisfied, then $h_3 \in C^{2\nu+1,\alpha}(\mathbb{R})$.

The discontinuities in h_3 are dealt with by means of a certain construction. If $\nu = -1$ let $d_0 = 1$. If ν is a non-negative integer satisfying $\nu \le \ell$ let $d_0, \cdots, d_{\nu+1}$ be the solution to the Vandermonde system

$$\sum_{\mu=0}^{\nu+1} d_\mu 2^{2k\mu} = \begin{cases} 0, & k = 0, \cdots, \nu, \\ 1, & k = \nu+1. \end{cases} \tag{2.9}$$

Let $b_{\nu+1}, \cdots, b_\ell$ be distinct positive numbers. Define

$$\zeta_j(y) = \sum_{\mu=0}^{\nu+1} d_\mu (\text{sgn } y) \exp\{-2^\mu b_j |y| / \sqrt{\varepsilon}\}, \; j = \nu+1, \cdots, \ell.$$

Thus

$$\zeta_j^{(2k)}(\pm 0) = \pm \varepsilon^{-k} b_j^{2k} \sum_{\mu=0}^{\nu+1} d_\mu 2^{2k\mu}.$$

Using (2.9),

$$\zeta_j^{(2k)}(\pm 0) = 0 \text{ for } k = 0, \cdots, \nu, \; j = \nu+1, \cdots, \ell. \tag{2.10}$$

Define a function ζ by $\zeta = \sum_{\nu+1}^{\ell} c_j \zeta_j$ where the $\ell - \nu$ numbers c_j, $j = \nu+1, \cdots, \ell$, are chosen so that

$$\zeta^{(2k)}(+0) = h_3^{(2k)}(+0) \text{ for } k = \nu+1, \cdots, \ell. \tag{2.11}$$

The equations (2.11) form a linear system of $\ell - \nu$ equations in the $\ell - \nu$ unknowns c_j. When written out, the equations (2.11) are

$$\sum_{j=\nu+1}^{\ell} c_j \zeta_j^{(2k)}(+0) = h_3^{(2k)}(+0) \text{ for } k = \nu+1, \cdots, \ell.$$

Inserting the formulas for the derivatives, these equations become

$$(-1)^k \left(\sum_{\mu=0}^{\nu+1} d_\mu 2^{2k\mu} \right) \sum_{j=\nu+1}^{\ell} c_j b_j^k = \varepsilon^k h_3^{(2k)}(+0) \text{ for } k = \nu+1, \cdots, \ell.$$

This gives a non-singular Vandermonde system for the c_j and, since $\varepsilon^k |h_3^{(2k)}(+0)| \le C(\bar{G}_{2\ell} + \bar{H}_{2\ell})$, the numbers c_j exist and satisfy

$$|c_j| \le C(\bar{G}_{2\ell} + \bar{H}_{2\ell}) \text{ for } j = \nu+1, \cdots, \ell.$$

Let z be the solution to the incoming half-plane problem

$$\begin{aligned} -\varepsilon\Delta z + pz_x + qz &= 0 \text{ for } x > 0, \\ z(0,y) &= \zeta(y) \text{ for } y \in \mathbb{R}. \end{aligned} \tag{2.12}$$

Let $u_4 = u_3 - z$. Then u_4 satisfies the incoming half-plane problem

$$(2.13)\qquad \begin{aligned} -\varepsilon\Delta u_4 + pu_{4,x} + qu_4 = 0 \text{ for } x > 0,\\ u_4(0,y) = h_4(y) := h_3(y) - \zeta(y) \text{ for } y \in \mathbb{R}. \end{aligned}$$

By our construction, $h_4 \in C^{2\ell,\alpha}(\mathbb{R})$ and

$$(2.14)\qquad |h_4^{(k)}(y)| \le C\varepsilon^{-k/2}e^{-c|y|/\sqrt{\varepsilon}}(\bar{G}_{2\ell} + \bar{H}_{2\ell}) \text{ for } k = 0, 1, \ldots, 2\ell.$$

Assembling the above functions, we have established the decomposition

$$(2.15)\qquad u = u_1 + u_2 = u_1 + u_3 = u_1 + u_4 + z \text{ in } \mathbb{Q}.$$

To obtain bounds for the derivatives of u we shall obtain bounds for the derivatives of each of the terms in the right-hand side (2.15). The function u_1 is given by the grazing half-plane problem (2.2) and bounds for its derivatives are obtained in Theorem 3.10. The function u_4 is given by the incoming half-plane problem (2.13). Its boundary data decays exponentially so u_4 has large y-derivatives near $y = 0$; its behaviour is analysed in Theorem 3.6. Although z is the solution of the incoming half-plane problem (2.12), the bounds for incoming half-plane problems given in Section 3 do not apply as the boundary data are discontinuous at $y = 0$. As has been remarked, these discontinuities correspond exactly to the incompatibilities in the data of the problem (2.1). Thus z contains the corner singular functions that are present in the solution of (2.1). The function z is analysed in Section 4. Bounds for the derivatives of u are established in Theorem 4.11.

3. Bounds on derivatives of solutions to half-plane problems

In this section we consider four boundary value problems on the half-planes $\Pi_x = \{(x,y) \in R^2 : x > 0\}$ and $\Pi_y = \{(x,y) \in R^2 : y > 0\}$. These problems arise both from the decomposition of u given in Section 2 and from a further decomposition that will appear in Section 5.

3.1. Growth conditions

A maximum principle will be used to bound the derivatives of the solution of each boundary-value problem. Since the half-plane is an unbounded domain, growth conditions are needed on the data for the solutions to exist and for the maximum principle to be satisfied. The derivation of these growth conditions is given in detail below for the incoming half-plane problem (the other problems are analogous): a Green's function for the problem is written in terms of modified Bessel functions of the first kind (cf. [10]), and then the desired growth condition, which merely ensures that the Green's function integrals defining the solution are finite, follows easily from standard properties of Bessel functions.

Consider the "incoming half-plane problem"

$$(3.1)\qquad Lv := -\varepsilon\Delta v + p_1 v_x + p_2 v_y + qv = f \text{ for } x > 0, \qquad v(0,y) = h(y) \text{ for } y \in (-\infty,\infty),$$

where p_1 and p_2 are unspecified constants. Set $v_1(x,y) = e^{-(p_1x+p_2y)/(2\varepsilon)}v(x,y)$. Then

$$-4\varepsilon^2\Delta v_1 + \kappa^2 v_1 = f_1(x,y) := 4\varepsilon e^{-(p_1x+p_2y)/(2\varepsilon)}f(x,y) \text{ for } x > 0, \quad v_1(0,y) = h_1(y) := h(y)e^{-p_2y/(2\varepsilon)},$$

with $\kappa^2 = p_1^2 + p_2^2 + 4\varepsilon q$ and $\eta = y/(2\varepsilon)$. Setting $\xi = x/(2\varepsilon)$, $\eta = y/(2\varepsilon)$, $v_2(\xi,\eta) = v_1(x,y)$, $f_2(\xi,\eta) = f_1(x,y)$, $h_2(\eta) = h_1(y)$, this becomes

$$(3.2)\qquad -\Delta v_2 + \kappa^2 v_2 = f_2(\xi,\eta) \text{ for } \xi > 0,\ v_2(0,\eta) = h_2(\eta).$$

The Green's function for this problem is easily verified to be

$$G(\xi,\eta;\sigma,\tau)=\frac{1}{2\pi}[K_0(\kappa\rho)-K_0(\kappa\rho_1)]$$

where $\rho=\sqrt{(\xi-\sigma)^2+(\eta-\tau)^2}$, $\rho_1=\sqrt{(\xi+\sigma)^2+(\eta-\tau)^2}$, and K_0 is a modified Bessel function of the second kind [1]. Hence with $\rho_2=\sqrt{\xi^2+(\eta-\tau)^2}$, the solution formula for (3.2) is

$$v_2(\xi,\eta)=\frac{1}{2\pi\kappa}\int_{\sigma=0}^{\infty}\int_{\tau=-\infty}^{\infty}f_2(\sigma,\tau)[K_0(\kappa\rho)-K_0(\kappa\rho_1)]d\tau d\sigma+\frac{\xi}{\pi}\int_{-\infty}^{\infty}h_2(\tau)\frac{1}{\rho_2}K_1(\kappa\rho_2)d\tau. \tag{3.3}$$

Returning to the original variables, the solution formula is

$$\begin{aligned}v(x,y)=&\frac{1}{2\pi\kappa\varepsilon}e^{(p_1x+p_2y)/(2\varepsilon)}\int_{s=0}^{\infty}\int_{t=-\infty}^{\infty}e^{-(p_1s+p_2t)/(2\varepsilon)}f(s,t)[K_0(\kappa r/(2\varepsilon))-K_0(\kappa r_1/(2\varepsilon))]dtds\\&+\frac{x}{2\pi\varepsilon}e^{(p_1x+p_2y)/(2\varepsilon)}\int_{-\infty}^{\infty}e^{-p_2t/(2\varepsilon)}h(t)\frac{1}{r_2}K_1(\kappa r_2/(2\varepsilon))dt,\end{aligned} \tag{3.4}$$

where $r=\sqrt{(x-s)^2+(y-t)^2}$, $r_1=\sqrt{(x+s)^2+(y-t)^2}$ and $r_2=\sqrt{x^2+(y-t)^2}$.

If the functions f_2 and h_2 are such that the integrals in (3.3) are convergent, then the solution v_2 of (3.2) exists and is given by (4). Furthermore, if f_2 and h_2 are non-negative, then v_2 is non-negative. Recalling that $|K_j(t)|\le Ct^{-1/2}e^{-t}$ for $t>1$, this leads to the following integrability conditions on f_2 and h_2 for the existence of a solution v_2, and hence for the maximum principle:

$$\int_{\sigma=1}^{\infty}\int_{|\tau|>1}(\sigma^2+\tau^2)^{-1/4}|f_2(\sigma,\tau)|e^{-\kappa(\sigma^2+\tau^2)^{1/2}}d\sigma d\tau<\infty,$$

$$\int_{|\tau|>1}|\tau|^{-3/2}|h_2(\tau)|e^{-\kappa|\tau|}d\tau<\infty.$$

In terms of f and h, and setting $r=(x^2+y^2)^{1/2}$, these conditions become

$$\begin{aligned}&\int_{x=1}^{\infty}\int_{|y|>1}r^{-1/2}|f(x,y)|e^{-(\kappa r+p_1x+p_2y)/(2\varepsilon)}dxdy<\infty,\\&\int_{y>1}|y|^{-3/2}|h(y)|e^{-(\kappa|y|+p_2y)/(2\varepsilon)}dy<\infty.\end{aligned} \tag{3.5}$$

3.2. Incoming half-plane problems

We shall consider two incoming half-plane boundary-value problems. First, based on (3.5), one has the following statement of conditions for the maximum principle for the operator L on Π_x:

Lemma 3.1. *Set $r=\sqrt{x^2+y^2}$. Let $\Phi\in C^2(\bar{\Pi}_x)$ satisfy $L\Phi\ge 0$ on Π_x, $\Phi(0,y)\ge 0$ for $y\in\mathbb{R}$ and*

$$\int_{x=0}^{\infty}\int_{y=-\infty}^{\infty}(1+r)^{-1/2}L\Phi(x,y)e^{-pr/(2\varepsilon)}dxdy<\infty,$$

$$\int_{-\infty}^{\infty}(1+|y|)^{-3/2}\Phi(0,y)e^{-p|y|/(2\varepsilon)}dy<\infty.$$

Then $\Phi(x,y)>0$ in Π_x. If $U\in C^2(\bar{\Pi}_x)$ satisfies $|LU(x,y)|\le L\Phi(x,y)$ in Π_x and $|U(0,y)|\le\Phi(0,y)$ for $y\in\mathbb{R}$, then $|U(x,y)|\le\Phi(x,y)$ in Π_x.

It can easily be checked that the barrier functions used in this sub-section satisfy these growth conditions.

The first incoming half-plane problem comes from Section 5. Let $U(x,y)$ be defined on $\bar{\Pi}_x$ by

$$LU=f^* \text{ on } \Pi_x,\quad U(0,y)=0 \text{ for all } y, \tag{3.6}$$

where f^* is smooth. We seek bounds on the derivatives of U. A problem such as (3.6) has no outflow or characteristic boundaries, and since the boundary $x=0$ and data f^* are smooth, one expects that all derivatives of U are bounded independently of ε. This will be shown rigourously in the next few Lemmas.

Lemma 3.2. *Let n be a non-negative integer. Let $f^* \in W^{n,\infty}(\Pi_x)$. Then $\|D_y^n U\|_{\infty,\Pi_x} \le \|f^*\|_{n,\infty,\Pi_x}/q$.*

Proof. From the boundary conditions, $D_y^n U(0,y) = 0$ for all y. Differentiating (3.6), $L(D_y^n U) = D_y^n f^*$ on Π_x. Set $w(x,y) = \|D_y^n f^*\|_{\infty,\Pi_x}/q$ on Π_x. Then $w(0,y) \ge |D_y^n(0,y)|$ for all y and $Lw = \|D_y^n f^*\|_{\infty,\Pi_x} \ge |D_y^n f^*|$ on Π_x. Thus w is a barrier function for $D_y^n U$, and the result follows from Lemma 3.1. ∎

Lemma 3.3. *Let n be a non-negative integer. Let $f^* \in W^{n+1,\infty}(\Pi_x)$. Then there exists a constant C such that $\|D_x D_y^n U\|_{\infty,\Pi_x} \le C\|f^*\|_{n+1,\infty,\Pi_x}$.*

Proof. Set $w(x,y) = x\|D_y^n f^*\|_{\infty,\Pi_x}/p$ for all $(x,y) \in \Pi_x$. Then $w(0,y) = 0 = |D_y^n U(0,y)|$ for all y. Also

$$Lw(x,y) = (1 + xq/p)\|D_y^n f^*\|_{\infty,\Pi_x} \ge \|D_y^n f^*\|_{\infty,\Pi_x} \ge |D_y^n f^*(x,y)| = L(D_y^n u)(x,y) \quad \text{on } \Pi_x.$$

Thus w is a barrier function for $D_y^n U$, and it follows from Lemma 3.1 that $|D_y^n U(x,y)| \le x\|D_y^n f^*\|_{\infty,\Pi_x}/p$ on $\bar{\Pi}_x$. As $D_y^n U(0,y) = 0$ for all y, this inequality implies that

$$|D_x D_y^n U(0,y)| = \left| \lim_{x\to 0^+} \frac{D_y^n U(x,y) - D_y^n U(0,y)}{x} \right| \le \|D_y^n f^*\|_{\infty,\Pi_x}/p \quad \text{for all } y > 0. \tag{3.7}$$

But $|L(D_x D_y^n U)| = |D_x D_y^n f^*| \le \|D_x D_y^n f^*\|_{\infty,\Pi_x}$ on Π_x. Using this inequality and (3.7), one can use a constant barrier function and Lemma 3.1 to get $|D_x D_y^n U(x,y)| \le p^{-1}\|D_y^n f^*\|_{\infty,\Pi_x} + q^{-1}\|D_x D_y^n f^*\|_{\infty,\Pi_x}$ on $\bar{\Pi}_x$, and the Lemma follows. ∎

Given a differential operator $D = D_x^m D_y^n$, set $|D| = m + n$.

Lemma 3.4. *Let m and n be non-negative integers. Set $D \equiv D_x^m D_y^n$. Let $f^* \in W^{|D|+2,\infty}(\Pi_x)$. Then there exists a constant C such that*

$$\|D_x^2 DU\|_{\infty,\Pi_x} \le C(\|f^*\|_{|D|+2,\infty,\Pi_x} + \|DU\|_{\infty,\Pi_x} + \|D_y^2 DU\|_{\infty,\Pi_x}).$$

Proof. Let

$$\tilde{U}(x,y) = DU(x,y) - DU(0,y) - xp^{-1}[Df^*(0,y) - qDU(0,y)].$$

Then $\tilde{U}(0,y) = 0$ for all y and

$$\begin{aligned}
L\tilde{U}(x,y) &= Df^*(x,y) + \varepsilon D_y^2 DU(0,y) - qDU(0,y) + \varepsilon xp^{-1}[D_y^2 Df^*(0,y) - qD_y^2 DU(0,y)] \\
&\quad - [Df^*(0,y) - qDU(0,y)] - xqp^{-1}[Df^*(0,y) - qDU(0,y)] \\
&= [Df^*(x,y) - Df^*(0,y)] + \varepsilon D_y^2 DU(0,y) + \varepsilon xp^{-1}[D_y^2 Df^*(0,y) - qD_y^2 DU(0,y)] \\
&\quad - xqp^{-1}[Df^*(0,y) - qDU(0,y)],
\end{aligned}$$

which implies that

$$\begin{aligned}
|L\tilde{U}(x,y)| &\le x\|f^*\|_{|D|+1,\infty,\Pi_x} + \varepsilon\|D_y^2 DU\|_{\infty,\Pi_x} + \varepsilon xp^{-1}(1+q)(\|D_y^2 Df^*\|_{\infty,\Pi_x} + \|D_y^2 DU\|_{\infty,\Pi_x}) \\
&\quad + xqp^{-1}(1+q)(\|Df^*\|_{\infty,\Pi_x} + \|DU\|_{\infty,\Pi_x}).
\end{aligned}$$

But $L(2\varepsilon p^{-1}x + x^2) = 2\varepsilon p^{-1}(p + qx) - 2\varepsilon + 2px + qx^2 \ge 2px$. Consequently we can choose a constant C_1 sufficiently large and independent of ε, U and f^* in such a way that the function

$$\psi(x,y) = C_1\left[\varepsilon x\|D_y^2 DU\|_{\infty,\Pi_x} + (2\varepsilon p^{-1}x + x^2)(\|f^*\|_{|D|+2,\infty,\Pi_x} + \|DU\|_{\infty,\Pi_x} + \|D_y^2 DU\|_{\infty,\Pi_x})\right] \tag{3.8}$$

is a barrier function for $\tilde{U}(x,y)$. By Lemma 3.1

$$|\tilde{U}(x,y)| \le \psi(x,y) \quad \text{for all } (x,y) \in \bar{\Pi}_x. \tag{3.9}$$

Now for all y,

$$DU_x(0,y) = \lim_{x\to 0+} \frac{DU(x,y) - DU(0,y)}{x} = \lim_{x\to 0+} \frac{\tilde{U}(x,y) + xp^{-1}[Df^*(0,y) - qDU(0,y)]}{x},$$

which by (3.8) and (3.9) implies that

$$\begin{aligned} |DU_x(0,y) - &p^{-1}[Df^*(0,y) - qDU(0,y)]| \\ &\le C_1\left[\varepsilon\|D_y^2 DU\|_{\infty,\Pi_x} + 2\varepsilon p^{-1}(\|f^*\|_{|D|+2,\infty,\Pi_x} + \|DU\|_{\infty,\Pi_x} + \|D_y^2 DU\|_{\infty,\Pi_x})\right]. \end{aligned} \tag{3.10}$$

But $L(DU) = Df^*$, so $(-\varepsilon D_x^2 DU - \varepsilon D_y^2 DU + pD_x DU + qDU)(0,y) = Df^*(0,y)$ for all y. Invoking (3.10) now shows that $|(\varepsilon D_x^2 DU + \varepsilon D_y^2 DU)(0,y)| \le C_1 \varepsilon\left[p\|D_y^2 DU\|_{\infty,\Pi_x} + 2(\|f^*\|_{|D|+2,\infty,\Pi_x} + \|DU\|_{\infty,\Pi_x} + \|D_y^2 DU\|_{\infty,\Pi_x})\right]$. It follows that

$$|D_x^2 DU(0,y)| \le C_2\|D_y^2 DU\|_{\infty,\Pi_x} + 2C_1(\|f^*\|_{|D|+2,\infty,\Pi_x} + \|DU\|_{\infty,\Pi_x}) \quad \text{for all } y, \tag{3.11}$$

where $C_2 = 1 + (2+p)C_1$.

Furthermore, $L(D_x^2 DU)(x,y) = D_x^2 Df^*(x,y)$ for all $(x,y) \in \Pi_x$. From this identity and (3.11), one can use a constant barrier function and Lemma 3.1 to get

$$|(D_x^2 DU)(x,y)| \le C_2\|D_y^2 DU\|_{\infty,\Pi_x} + 2C_1(\|f^*\|_{|D|+2,\infty,\Pi_x} + \|DU\|_{\infty,\Pi_x}) + q^{-1}\|D_x^2 Df^*\|_{\infty,\Pi_x} \text{ on } \bar{\Pi}_x,$$

which proves the desired result. ∎

Finally we combine the previous three Lemmas in the following definitive result.

Theorem 3.5. *Let m and n be non-negative integers. Let $f^* \in W^{m+n,\infty}$. Then there exists a constant C such that*

$$\|D_x^m D_y^n U\|_{\infty,\Pi_x} \le C\|f^*\|_{m+n,\infty,\Pi_x}.$$

Proof. We use induction on m. The cases $m = 0, 1$ are proved in Lemmas 3.2 and 3.3. Let k be a positive integer. Assume that the Theorem holds true for $m = 0, 1, \ldots, k$. Let n be a non-negative integer. Applying Lemma 3.4 with $D = D_x^{k-1} D_y^n$ and invoking the inductive hypothesis yields

$$\|D_x^{k+1} D_y^n U\|_{\infty,\Pi_x} \le C(\|f^*\|_{k+n+1,\infty,\Pi_x} + \|DU\|_{\infty,\Pi_x} + \|D_y^2 DU\|_{\infty,\Pi_x}) \le C\|f^*\|_{k+n+1,\infty,\Pi_x}.$$

That is, the Theorem holds true when $m = k+1$. By induction we are done. ∎

The second incoming half-plane problem defines u_4 in (2.13). That is, we seek bounds on the derivatives of the solution of the boundary-value problem

$$Lu_4 = 0 \text{ on } \Pi_x, \quad u_4(0,y) = h_4(y) \text{ for } y \in \mathbb{R}, \tag{3.12}$$

where $h_4 \in C^{2\ell,\alpha}(\mathbb{R})$ is an odd function of y and

$$|h_4^{(k)}(y)| \le C\varepsilon^{-k/2} e^{-\sqrt{q}|y|/(2\sqrt{\varepsilon})} \quad \text{for } k = 0, 1, \ldots, 2\ell. \tag{3.13}$$

In (3.13) the boundary data behaves like a characteristic boundary layer sampled at $x = 0$, and one expects that this layer behaviour will be convected downstream; furthermore, there is no other reason to expect layer behaviour in the solution u_4. The particular assumption (3.13) comes from Theorem 3.10, because in (2.13) part of the data h_4 comes from the solution of a grazing half-plane problem. While (3.6) is also an incoming half-plane problem, its solution U contains no layers and so is quite different in nature from u_4.

Define $\phi_1(x,y) = \exp(-qx/(2p))\exp(-\sqrt{q}y/(2\sqrt{\varepsilon}))$. Then on Π_x,

$$L\phi_1(x,y) = A_2\phi_1(x,y), \quad \text{where } A_2 = -\varepsilon(q/(2p))^2 - q/4 - q/2 + q > 0 \text{ for } \varepsilon < p^2/q. \tag{3.14}$$

This inequality implies that

$$L(x\phi_1)(x,y) = xL\phi_1(x,y) - 2\varepsilon(\phi_1)_x(x,y) + p\phi_1(x,y) \ge p\phi_1(x,y) \text{ on } \Pi_x. \tag{3.15}$$

These barrier functions will be used in the following result, which bounds the derivatives of u_4.

Theorem 3.6. *Let $\varepsilon < p^2/q$. Then there exists a constant C such that for all $(x,y) \in \bar{\Pi}_x$,*

$$|D_x^m D_y^n u_4(x,y)| \le C\varepsilon^{-n/2}\phi_1(x,|y|) \text{ for } m \ge 0,\ n \ge 0 \text{ and } 2m+n \le 2\ell. \tag{3.16}$$

Proof. The data of the problem imply that $u_4 \in C^{2\ell,\alpha}(\bar{\Pi}_x)$.

We use induction on m to prove first that for all $(x,y) \in \bar{\Pi}_x$,

$$|D_x^m D_y^n u_4(x,y)| \le C\varepsilon^{-n/2}\phi_1(x,y) \text{ for } m \ge 0,\ n \ge 0 \text{ and } 2m+n \le 2\ell. \tag{3.17}$$

If $m = 0$, then by (3.13) one can choose a constant C such that for $n \le 2\ell$ one has $|h_4^{(n)}(y)| \le C\varepsilon^{n/2}\phi_1(0,y)$ for all $y \ge 0$. Now (3.14) implies that

$$L\big(C\varepsilon^{n/2}\phi_1(x,y)\big) > 0 = L\big(D_y^n u_4(x,y)\big) \text{ on } \Pi_x.$$

Thus $C\varepsilon^{n/2}\phi_1$ is a barrier function for $D_y^n u_4$ on Π_x and the case $m = 0$ is complete.

Next, assume that (3.17) holds true for $m = k$, where k is some non-negative integer, and all n satisfying $2k + n \le 2\ell - 2$. Let n be a fixed non-negative integer with $2(k+1) + n \le 2\ell$. Set

$$\tilde{u}_4(x,y) = D_x^k D_y^n u_4(x,y) - D_x^k D_y^n u_4(0,y).$$

Then $\tilde{u}_4(0,y) = 0$ for all y and (3.12) implies that $L\tilde{u}_4(x,y) = \varepsilon D_x^k D_y^{n+2} u_4(0,y) - q D_x^k D_y^n u_4(0,y)$. By the inductive hypothesis, for $2k + n + 2 \le 2\ell$ we get

$$|L\tilde{u}_4(x,y)| \le C\varepsilon^{-n/2}\phi_1(x,y).$$

Hence, by (3.15), one can choose a barrier function to prove $|\tilde{u}_4(x,y)| \le C\varepsilon^{-n/2}x\phi_1(x,y)$ on Π_x provided that $2k + n + 2 \le 2\ell$. Thus for all $y \ge 0$,

$$\big|D_x^{k+1} D_y^n u_4(0,y)\big| = \left|\lim_{x\to 0^+} \frac{\tilde{u}_4(x,y)}{x}\right| \le C\varepsilon^{n/2}\phi_1(0,y).$$

From (3.12), $L(D_x^{k+1} D_y^n u_4)(x,y) = 0$. Recalling (3.14), it is now clear that one can construct a barrier function showing that $|D_x^{k+1} D_y^n v(x,y)| \le C\varepsilon^{n/2}\phi_1(x,y)$ on Π_x provided that $2k+n+2 \le 2\ell$. That is, (3.17) holds true with $m = k+1$ and the induction is complete.

The bound of (3.17) implies (3.16) for $y \ge 0$. But the function $h_4(y)$ is odd, which implies that $u_4(x,y)$ is an odd function of y. It follows that (3.16) holds true also for $y \le 0$, which completes the proof. ■

The function ϕ_1 decays rapidly away from $y = 0$, so (3.16) shows that all layer-type behaviour in u_4 occurs in a narrow region immediately downstream of that small portion of the y-axis where $h_4(y)$ changes rapidly. This is consistent with our intuition.

3.3. Grazing half-plane problem

Next, consider the "grazing half-plane" problem that defines the function u_1 in (2.3). The associated conditions for a maximum principle on Π_y are stated in the next Lemma.

Lemma 3.7. *Set $r = \sqrt{x^2+y^2}$. Let $\Phi \in C^2(\bar{\Pi}_y)$ satisfy $L\Phi \ge 0$ on Π_y, $\Phi(x,0) \ge 0$ for $x \in \mathbb{R}$ and*

$$\int_{x=0}^{\infty}\int_{y=-\infty}^{\infty} (1+r)^{-1/2} L\Phi(x,y) e^{-\kappa r/(2\varepsilon)}\,dx dy < \infty,$$

$$\int_{x=-\infty}^{0}\int_{y=-\infty}^{\infty} (1+r)^{-1/2} L\Phi(x,y) e^{-q\kappa^{-1} r}\,dx dy < \infty,$$

$$\int_{0}^{\infty} (1+x)^{-3/2}\Phi(x,0) e^{-(\kappa+p)x/(2\varepsilon)}\,dx < \infty,$$

$$\int_{-\infty}^{0} (1+|x|)^{-3/2}\Phi(x,0) e^{-q\kappa^{-1}|x|}\,dx < \infty,$$

where $\kappa = \sqrt{p^2 + 4\varepsilon q}$. *Then* $\Phi(x,y) > 0$ *in* Π_y. *If* $U \in C^2(\bar{\Pi}_y)$ *satisfies* $|LU(x,y)| \le L\Phi(x,y)$ *in* Π_y *and* $|U(x,0)| \le \Phi(x,0)$ *for* $x \in \mathbb{R}$, *then* $|U(x,y)| \le \Phi(x,y)$ *in* Π_y.

Our concern is derivative bounds for the solution of the following problem:

$$Lu_1 = 0 \text{ on } \Pi_y, \quad u_1(x,0) = g_1(x) \text{ for all } x, \tag{3.18}$$

where $g_1 \equiv 0$ for $x \le -1$, $|g_1^{(i)}(x)| \le Ce^{-a_1 x}$ for $x \ge -1$ and $i = 0, 1, \ldots, 2\ell$; here a_1 is a positive constant that satisfies $a_1 \ge q/(2p)$.

For $y > 0$ the solution of the reduced problem in (3.18) is the function 0, which is in general inconsistent with the boundary data at $y = 0$, so we expect u_1 to have a characteristic boundary layer along $y = 0$.

Lemma 3.8. *Suppose that* $0 \le m \le 2\ell$. *Then there exists a constant* C *such that*

$$|D_x^m u_1(x,y)| \le C\|g_1\|_{m,\infty,\mathbb{R}}\phi_1(x,y) \text{ on } \bar{\Pi}_y. \tag{3.19}$$

Proof. Now $L(\|g_1\|_{0,\infty,\mathbb{R}}\phi_1(x,y)) = A_2\|g_1\|_{0,\infty,\mathbb{R}}\phi_1(x,y) \ge 0 = Lu_1(x,y)$ on Π_y, and $|u_1(x,0)| = |g_1(x)| \le C\|g_1\|_{0,\infty,\mathbb{R}}\phi_1(x,0)$ since $a_1 \ge q/(2p)$. A barrier function argument shows immediately that $|u_1(x,y)| \le C\|g_1\|_{\infty,\mathbb{R}}\phi_1(x,y)$ on $\bar{\Pi}_y$. That is, (3.19) holds true when $m = 0$.

For $m > 0$, the function $D_x^m u_1$ is the solution of a problem similar to (3.18) but with boundary data $g_1^{(m)}(x)$; applying Lemma 3.8 with $m = 0$ to this problem yields (3.19) with $m > 0$. ∎

For each integer n, let $\bar{n}$ denote the smallest even integer that satisfies $\bar{n} \ge n$.

Lemma 3.9. *There exists a constant* C *such that for* $n = 1, 2$ *and* $m = 0, 1, \ldots, 2\ell - 2$,

$$|D_x^m D_y^n u_1(x,y)| \le C\varepsilon^{-n/2}\|g_1\|_{m+\bar{n},\infty,\mathbb{R}}\phi_1(x,y) \text{ on } \bar{\Pi}_y. \tag{3.20}$$

Proof. Solving the equation $Lu = 0$ for u_{yy} and invoking the cases $m = 0, 1, 2$ of Lemma 3.8 to bound the x-derivative terms, one gets easily

$$|D_y^2 u_1(x,y)| \le C\varepsilon^{-1}\|g_1\|_{2,\infty,\mathbb{R}}\phi_1(x,y) \text{ on } \bar{\Pi}_y.$$

This proves (3.20) with $m = 0$ and $n = 2$.

We deduce the case $m = 0$ and $n = 1$ by means of an interpolation argument (cf. [8]). Let $(x,y) \in \bar{\Pi}_y$ be arbitrary but fixed. By the mean value theorem there exists $y^* \in (y, y + \sqrt{\varepsilon})$ such that

$$|D_y u_1(x,y^*)| = |[u_1(x, y + \sqrt{\varepsilon}) - u_1(x,y)]/\sqrt{\varepsilon}| \le C\varepsilon^{-1/2}\|g_1\|_{0,\infty,\mathbb{R}}\phi_1(x,y),$$

where we used Lemma 3.8 and the fact that $\phi_1(x,y)$ is a decreasing function of y. Hence

$$\begin{aligned}
|D_y u_1(x,y)| &= \left| D_y u_1(x,y^*) + \int_{t=y^*}^{y} D_y^2(x,t)\,dt \right| \\
&\le C\varepsilon^{-1/2}\|g_1\|_{0,\infty,\mathbb{R}}\phi_1(x,y) + C\int_{t=y}^{y^*} \varepsilon^{-1}\|g_1\|_{2,\infty,\mathbb{R}}\phi_1(x,t)\,dt \\
&\le C\varepsilon^{-1/2}\|g_1\|_{2,\infty,\mathbb{R}}[\phi_1(x,y) + \phi_1(x,y^*)] \\
&\le C\varepsilon^{-1/2}\|g_1\|_{2,\infty,\mathbb{R}}\phi_1(x,y).
\end{aligned}$$

Lemma 3.9 has now been proved for the case $m = 0$. For $m > 0$, the function $D_x^m u_1$ is the solution of a problem similar to (3.18) but with boundary data $g_1^{(m)}$; applying the case $m = 0$ of the Lemma to this function yields (3.20). ∎

Theorem 3.10. *There exists a constant C such that for $n = 0, 1, \ldots, 2\ell + 1$ and $m = 0, 1, \ldots, 2\ell - \bar{n}$,*

$$|D_x^m D_y^n u_1(x, y)| \le C\varepsilon^{-n/2} \|g_1\|_{m+\bar{n},\infty,\mathbb{R}} \phi_1(x, y) \text{ on } \bar{\Pi}_y. \tag{3.21}$$

Proof. We use induction on n. For $n = 0, 1, 2$, the result is already proved in Lemmas 3.8 and 3.9. Fix an integer $k \ge 2$. Suppose that the result holds true for $n = 0, 1, \ldots, k$. Now $D_y^{k-1} L u_1 = 0$ yields

$$\begin{aligned} |\varepsilon D_y^{k+1} u_1(x, y)| &= |(-\varepsilon D_x^2 D_y^{k-1} u_1 + p D_x D_y^{k-1} u_1 + q D_y^{k-1} u_1)(x, y)| \\ &\le C\varepsilon^{-(k-1)/2} \big(\varepsilon \|g_1\|_{2+\overline{k-1},\infty,\mathbb{R}} + p\|g_1\|_{1+\overline{k-1},\infty,\mathbb{R}} + q\|g_1\|_{\overline{k-1},\infty,\mathbb{R}}\big) \phi_1(x, y), \end{aligned}$$

where we used the inductive hypothesis. Hence

$$|D_y^{k+1} u_1(x, y)| \le C\varepsilon^{-(k+1)/2} \|g_1\|_{\overline{k+1},\infty,\mathbb{R}}\, \phi_1(x, y).$$

For each m, the bound

$$|D_x^m D_y^{k+1} u_1(x, y)| \le C\varepsilon^{-(k+1)/2} \|g_1\|_{m+\overline{k+1},\infty,\mathbb{R}}\, \phi_1(x, y)$$

then follows in the usual way by considering the half-plane problem for which $D_x^m u_1$ is the solution. By the principle of induction the proof is complete. ■

3.4. Outgoing half-plane problem

Finally, Section 5 leads to the "outgoing half-plane" problem

$$MW := -\varepsilon \Delta W - pW_x + qW = 0 \text{ on } \Pi_x, \quad W(0, y) = W_0(y) \text{ for all } y. \tag{3.22}$$

Note that in the definition of the operator M the convection term has sign opposite to that in L, so the convective vector here is out of Π_x. The appropriate maximum principle for this problem is given in

Lemma 3.11. *Set $r = \sqrt{x^2 + y^2}$. Let $\Phi \in C^2(\bar{\Pi}_x)$ satisfy $M\Phi \ge 0$ on Π_x, $\Phi(0, y) \ge 0$ for $y \in \mathbb{R}$ and*

$$\begin{aligned} &\int_{x=0}^{\infty} \int_{y=-\infty}^{\infty} (1 + r)^{-1/2} M\Phi(x, y) e^{-q\kappa^{-1} r} dx dy < \infty, \\ &\int_{-\infty}^{\infty} (1 + |y|)^{-3/2} \Phi(0, y) e^{-\kappa|y|/(2\varepsilon)} dy < \infty, \end{aligned}$$

where $\kappa = \sqrt{p^2 + 4\varepsilon q}$. Then $\Phi(x, y) > 0$ in Π_x. If $W \in C^2(\bar{\Pi}_x)$ satisfies $|MW(x, y)| \le L\Phi(x, y)$ in Π_x and $|W(0, y)| \le \Phi(0, y)$ for $y \in \mathbb{R}$, then $|W(x, y)| \le \Phi(x, y)$ in Π_x.

The first result is applicable both to W and, subsequently, to other functions.

Lemma 3.12. *Suppose that $MZ(x, y) = 0$ on Π_x, $Z(0, y) = Z_0(y)$ for all y and $\|Z_0\|_{\infty,\mathbb{R}}$ is finite. Then*

$$|Z(x, y)| \le \|Z_0\|_{\infty,\mathbb{R}} e^{-px/\varepsilon} \quad \text{for all } (x, y) \in \bar{\Pi}_x.$$

Proof. Consider the function $\phi(x, y) = \|Z_0\|_{\infty,\mathbb{R}} e^{-px/\varepsilon}$. Then $\phi(0, y) = \|Z_0\|_{\infty,\mathbb{R}} \ge |Z(0, y)|$ for all y, and $M\phi(x, y) = q\|Z_0\|_{\infty,\mathbb{R}} e^{-px/\varepsilon} > 0 = |MZ(x, y)|$ for all $(x, y) \in \Pi_x$. The maximum principle of Lemma 3.11 yields $|Z(x, y)| \le \phi(x, y)$ on $\bar{\Pi}_x$. ■

Set $C_0 = 1,\ C_1 = (2p^{-1} + p)(1 + q)$, and $C_i = pC_{i-1} + (1 + q)C_{i-2}$ for $i = 2, 3, \ldots$.

Lemma 3.13. *If $\|W_0\|_{\infty,\mathbb{R}}$ is finite, then*

$$|W(x,y)| \le C_0\|W_0\|_{\infty,\mathbb{R}} e^{-px/\varepsilon} \quad \text{for all } (x,y)\in\bar{\Pi}_x. \tag{3.23}$$

If $\|W_0\|_{2,\infty,\mathbb{R}}$ is finite, then

$$|W_x(x,y)| \le C_1\|W_0\|_{2,\infty,\mathbb{R}}\varepsilon^{-1} e^{-px/\varepsilon} \quad \text{for all } (x,y)\in\bar{\Pi}_x. \tag{3.24}$$

Proof. Inequality (3.23) is immediate from Lemma 3.12. To prove (3.24), we must first bound $|W_x(0,y)|$. Set $\theta(x,y) = W(x,y) - W_0(y)e^{-px/\varepsilon}$ on $\bar{\Pi}_x$. Then $\theta(0,y)=0$ for all y and

$$|M\theta(x,y)| = \left|[\varepsilon W_0''(y) - qW_0(y)]e^{-px/\varepsilon}\right| \le (\varepsilon\|W_0''\|_{\infty,\mathbb{R}} + q\|W_0\|_{\infty,\mathbb{R}})e^{-px/\varepsilon}.$$

Set

$$\phi(x,y) = \frac{4\varepsilon}{p^2}(\varepsilon\|W_0''\|_{\infty,\mathbb{R}} + q\|W_0\|_{\infty,\mathbb{R}})\left(e^{-px/(2\varepsilon)} - e^{-px/\varepsilon}\right).$$

Then $\phi(0,y) = 0 = \theta(0,y)$ for all y, and

$$\begin{aligned} M\phi(X,Y) &= \frac{4\varepsilon}{p^2}(\varepsilon\|W_0''\|_{\infty,\mathbb{R}} + q\|W_0\|_{\infty,\mathbb{R}})\left[\frac{p^2}{4\varepsilon}e^{-px/(2\varepsilon)} + q\left(e^{-px/(2\varepsilon)} - e^{-px/\varepsilon}\right)\right] \\ &\ge (\varepsilon\|W_0''\|_{\infty,\mathbb{R}} + q\|W_0\|_{\infty,\mathbb{R}})e^{-px/(2\varepsilon)} \\ &\ge |M\theta(x,y)| \text{ for all } (x,y)\in\Pi_x. \end{aligned}$$

By Lemma 3.11, $|\theta(x,y)| \le \phi(x,y)$ for all $(x,y)\in\bar{\Pi}_x$. Consequently for all y we have

$$\begin{aligned} |W_x(0,y)| &= \left|\lim_{x\to 0^+}\frac{W(x,y)-W(0,y)}{x}\right| \\ &= \left|\lim_{x\to 0^+}\frac{\theta(x,y)+W_0(y)e^{-px/\varepsilon} - W_0(y)}{x}\right| \\ &\le \lim_{x\to 0^+}\frac{|\phi(x,y)|}{x} + |W_0(y)|\lim_{x\to 0^+}\left|\frac{e^{-px/\varepsilon}-1}{x}\right| \\ &= \frac{4\varepsilon}{p^2}(\|W_0''\|_{\infty,\mathbb{R}} + q\|W_0\|_{\infty,\mathbb{R}})\frac{p}{2\varepsilon} + \frac{p}{\varepsilon}|W_0(y)| \\ &\le C_1\varepsilon^{-1}\|W_0\|_{2,\infty,\mathbb{R}}. \end{aligned} \tag{3.25}$$

Apply Lemma 3.12 to W_x to get $|W_x(x,y)| \le C_1\|W_0\|_{2,\infty,\mathbb{R}}\varepsilon^{-1}e^{-px/\varepsilon}$ on $\bar{\Pi}_x$. ■

We can now give a bound on all derivatives of solutions of (3.22).

Theorem 3.14. *Let m and n be non-negative integers. Suppose that $\|W_0\|_{\bar{m}+n,\infty,\mathbb{R}}$ is finite. Then*

$$|D_x^m D_y^n W(x,y)| \le C_m\|W_0\|_{\bar{m}+n,\infty,\mathbb{R}}\,\varepsilon^{-m}e^{-px/\varepsilon} \text{ for all } (x,y)\in\Pi_x. \tag{3.26}$$

Proof. We first prove the result using strong induction on m under the assumption that $n=0$. The cases $m=0,1$ have been dealt with in Lemma 3.13.

Assume that the Theorem (with $n=0$) holds true for $m=0,1,\dots,k$, where k is some positive integer. Suppose that $\|W_0\|_{\overline{k+1},\infty,\mathbb{R}}$ is finite. The function W_{yy} satisfies $MW_{yy}=0$ on Π_x and $W_{yy}(0,y) = W_0''(y)$ for all Y . That is, W_{yy} is the solution of a problem similar to (3.22) but with boundary data W_0''. Observe that $\|W_0''\|_{\overline{k-1},\infty,\mathbb{R}}$ is finite. By the inductive hypothesis (with $m=k-1$) applied to W_{yy},

$$|D_x^{k-1}W_{yy}(x,y)| \le C_{k-1}\|W_0''\|_{\overline{k-1},\infty,\mathbb{R}}\,\varepsilon^{-k+1}e^{-px/\varepsilon} \le C_{k-1}\|W_0\|_{\overline{k+1},\infty,\mathbb{R}}\,\varepsilon^{-k+1}e^{-px/\varepsilon} \tag{3.27}$$

for all $(x,y) \in \bar{\Pi}_x$. When the equation $MW = 0$ is differentiated $k-1$ times with respect to x, one gets

$$\begin{aligned} |\varepsilon D_x^{k+1} W(x,y)| &= |(-\varepsilon D_x^{k-1} D_y^2 W - p D_x^k W + q D_x^{k-1} W)(x,y)| \\ &\le \left(C_{k-1}\|W_0\|_{\overline{k+1},\infty,\mathbb{R}}\varepsilon^{-k+2} + pC_k\|W_0\|_{\overline{k},\infty,\mathbb{R}}\varepsilon^{-k} + qC_{k-1}\|W_0\|_{\overline{k-1},\infty,\mathbb{R}}\varepsilon^{-k+1}\right)e^{-px/\varepsilon} \\ &\le C_{k+1}\|W_0\|_{\overline{k+1},\infty,\mathbb{R}}\,\varepsilon^{-k}e^{-px/\varepsilon} \text{ for all } (x,y) \in \bar{\Pi}_x, \end{aligned}$$

by (3.27), the inductive hypothesis, and the definition of C_{k+1}. This proves the result for the case $m = k+1$. By the principle of induction, the proof is complete for the case $n = 0$.

If $n > 0$, then apply the result just proved to the function $D_y^n W$, which satisfies a problem similar to (3.22) but with boundary data $W_0^{(n)}$. ■

4. Bounds for z

In this section we obtain bounds for the derivatives of the function z that appears in the expansion (2.12). The bounds are obtained by working directly with the solution formula for the incoming half-plane problem that defines z. As a consequence of these bounds, bounds for the solution of (2.1) are derived at the end of the section.

From (2.10), z is the solution to the problem $Lz = 0$ with boundary data $\zeta = \sum_j c_j \zeta_j$. By linearity it suffices to study the problem $Lz = 0$ with boundary data ζ_j. Fixing j and using the notation $\hat{b}_\mu = 2^\mu b_j$, ζ_j is given by

$$\zeta_j(y) = \sum_{\mu=0}^{\nu+1} d_\mu (\text{sgn } y) e^{-\hat{b}_\mu |y|/\sqrt{\varepsilon}}.$$

We shall use the following notation: z_μ is the solution to the following half-plane problem: $Lz_\mu = 0$ on Π_x, $z_\mu(0,y) = (\text{sgn } y)\exp(-\hat{b}_\mu|y|/(2\sqrt{\varepsilon}))$ for $y \neq 0$. Thus $z(x,y) = \sum_{\mu=0}^{\nu+1} d_\mu z_\mu(x,y)$ on Π_x. From (2.7) the numbers d_μ satisfy

$$\sum_{\mu=0}^{\nu+1} d_\mu \hat{b}_\mu^{2k} = 0 \text{ for } k = 0, \cdots, \nu. \tag{4.1}$$

Equations (4.1), which are an expression of the compatibility satisfied by the data of (2.1), will be used in what follows. The value $\nu = -1$ signifies that g satisfies no compatibility; in this case, the d_μ satisfy no linear relations.

We seek a bound on all derivatives of z on the half-plane $\Pi_x = \{(x,y) \in R^2 : x > 0\}$. Our methodology is the following: first bound z (Lemmas 4.1 and 4.2), then differentiate an integral representation of z to bound all its even-order y-derivatives (Lemmas 4.5 and 4.7), use this result to bound the odd-order z-derivatives (Lemma 4.9), and finally invoke these bounds in a Kellogg and Tsan-type argument that bounds all mixed derivatives of z (Theorem 4.10). In the final result (Theorem 4.11) we obtain derivative bounds for the solution of the quarter-plane problem (2.1).

We start with the solution formula for z_μ. From (3.4) with $f \equiv 0$, $p_1 = p$, $p_2 = 0$ and $h(y) = (\text{sgn } y)\exp(-\hat{b}_\mu|y|/(2\sqrt{\varepsilon}))$,

$$z_\mu(x,y) = \frac{x}{2\varepsilon\pi} e^{px/(2\varepsilon)} \int_{-\infty}^{\infty} (\text{sgn}\,\tau) e^{-\hat{b}_\mu|t|/(2\sqrt{\varepsilon})} \frac{1}{r_2} K_1(\kappa r_2/(2\varepsilon))dt, \tag{4.2}$$

where $r_2 = \sqrt{x^2 + (y-t)^2}$ and the functions $\{K_j\}_{j=1}^\infty$ are the standard modified Bessel functions of the first kind that vanish at infinity [1, §9.6]. Since $z_\mu(0,y)$ is discontinuous at $y = 0$ the maximum principle cannot be used directly to obtain bounds for z_μ. However bounds are readily obtained, for $r \le \varepsilon$, from (4.2).

Lemma 4.1. *For all $(x,y) \in \Pi_x$ with $r \le \varepsilon$, we have $|z_\mu(x,y)| \le C$ for each μ.*

Proof. Since $x \le \varepsilon$, (4.2) implies that

$$|z_\mu(x,y)| \le \frac{Cx}{\varepsilon}\left(\int_{|t-y|\le\varepsilon} + \int_{|t-y|>\varepsilon}\right)\frac{1}{r_2}K_1(\kappa r_2/(2\varepsilon))\,dt = J_1 + J_2, \quad \text{say.} \tag{4.3}$$

To estimate J_1, observe that $x \le \varepsilon$ and $|t-y| \le \varepsilon$ imply that $\kappa r_2/(2\varepsilon) \le C$, so $K_1(\kappa r_2/(2\varepsilon)) \le C\varepsilon/r_2$ (see [1,(9.6.9]). Hence

$$J_1 \le \frac{Cx}{\varepsilon}\int_{|t-y|\le\varepsilon}\frac{\varepsilon}{r_2^2}\,dt = C\int_{|t-y|\le\varepsilon}\frac{x}{x^2+(t-y)^2}\,dt \le C, \tag{4.4}$$

as the indefinite integral here is $C\arctan((t-y)/x)$ and $|\arctan(\cdot)| \le \pi/2$. In J_2, $|t-y| > \varepsilon$ implies that $\kappa r_2/(2\varepsilon) \ge \kappa/2 \ge p/2 > 0$, so [1,(9.7.2)]

$$K_1(\kappa r_2/(2\varepsilon)) \le C(\varepsilon/r_2)^{1/2}e^{-\kappa r_2/(2\varepsilon)} \le C(\varepsilon/r_2)^{1/2}e^{-\kappa|t-y|/(2\varepsilon)}.$$

Hence

$$J_2 \le \frac{Cx}{\sqrt{\varepsilon}}\int_{|t-y|>\varepsilon}\frac{1}{r_2^{3/2}}e^{-\kappa|t-y|/(2\varepsilon)}\,dt. \tag{4.5}$$

By the inequality $2ab \le a^2+b^2$,

$$\frac{1}{r_2^{3/2}} = \frac{1}{r_2}\cdot\frac{1}{r_2^{1/2}} \le \frac{1}{r_2}\left(\frac{\sqrt{\varepsilon}}{2r_2}+\frac{1}{2\sqrt{\varepsilon}}\right) = \frac{1}{2}\left(\frac{\sqrt{\varepsilon}}{r_2^2}+\frac{1}{r_2\sqrt{\varepsilon}}\right).$$

Hence

$$J_2 \le C\int_{|t-y|>\varepsilon}\frac{x}{x^2+(t-y)^2}\,dt + C\varepsilon^{-1}\int_{|t-y|>\varepsilon}\frac{x}{r_2}e^{-\kappa|t-y|/(2\varepsilon)}\,dt \le C, \tag{4.6}$$

since the first indefinite integral is $C\arctan((t-y)/x)$ and in the second integral $x/r_2 \le 1$, after which this integral can be evaluated exactly. Combining (4.3), (4.4) and (4.6), the proof is complete. ∎

From §3, recall the function $\phi_1(x,y) = \exp(-qx/(2p))\exp(-\sqrt{q}y/(2\sqrt{\varepsilon}))$. Define

$$\phi_2(x,y) = \exp(-qx/(2p))\exp(-\beta y/(2\sqrt{\varepsilon})),$$

where $\beta = \min\{\sqrt{q}, \hat{b}_0, \cdots, \hat{b}_{\nu+1}\}$. Then $L\phi_2(x,y) = A_4\phi_2(x,y)$, where $A_4 = -\varepsilon(q/(2p))^2 - \beta^2/4 - q/2 + q > 0$ for $\varepsilon < p^2/q$. For each $\rho > 0$, set

$$\Pi_{x,\rho} = \{(x,y) \in R^2 : x > 0,\ \sqrt{x^2+y^2} > \rho\} \text{ and } Q_\rho = \{(x,y) \in R^2 : x > 0,\ y > 0,\ \sqrt{x^2+y^2} > \rho\}.$$

Lemma 4.2. *There exists a constant C such that for all $(x,y) \in \Pi_x$, $|z_\mu(x,y)| \le C\phi_2(x,|y|)$ for $\mu = 0,1,\dots,\nu+1$ and $|z(x,y)| \le C\phi_2(x,|y|)$.*

Proof. The bound on $|z|$ follows immediately from the bound on the $|z_\mu|$. Fix μ. Lemma 4.1 implies that $|z_\mu(x,y)| \le C\phi_2(x,|y|)$ for $\sqrt{x^2+y^2} \le \sqrt{\varepsilon}$, so it remains only to prove this inequality on $\bar{\Pi}_{x,\sqrt{\varepsilon}}$. The boundary conditions for z_μ form an odd function of y, and it then follows from $Lz_\ell = 0$ that $z_\mu(x,y)$ is an odd function of y. Thus to estimate $|z_\mu(x,y)|$ on $\bar{\Pi}_{x,\sqrt{\varepsilon}}$, it is enough to estimate $|z_\mu(x,y)|$ on $\bar{Q}_{\sqrt{\varepsilon}}$. As z_μ is continuous and odd on Π_x, $z_\mu(x,0) = 0$ for all $x > 0$. Now z_μ is defined on $Q_{\sqrt{\varepsilon}}$ by $Lz_\mu = 0$ and certain data on the boundary of $Q_{\sqrt{\varepsilon}}$: on the curved part of $\partial Q_{\sqrt{\varepsilon}}$ we know only that $|z_\mu| \le 1$, while on the two straight parts we have $z_\mu(x,0) = 0$ and the boundary data $z_\mu(0,y) = \exp(-\hat{b}_\mu y/(2\sqrt{\varepsilon}))$ for $y \ge \sqrt{\varepsilon}$. Thus we can choose C so that $C\phi_2$ is a barrier function for $\pm z_\mu$ on $\bar{Q}_{\sqrt{\varepsilon}}$, and the desired result follows. ■

We now prepare for the derivative bounds. Let $s = t - y$ in (4.2). Then

$$z_\mu(x,y) = \frac{x}{2\pi\varepsilon}\, e^{px/(2\varepsilon)} \left[-\int_{s=-\infty}^{-y} e^{\hat{b}_\mu(s+y)/(2\sqrt{\varepsilon})} \frac{1}{\sqrt{x^2+s^2}} K_1\left(\frac{\kappa\sqrt{x^2+s^2}}{2\varepsilon}\right) ds \right.$$
$$\left. + \int_{s=-y}^{\infty} e^{-\hat{b}_\mu(s+y)/(2\sqrt{\varepsilon})} \frac{1}{\sqrt{x^2+s^2}} K_1\left(\frac{\kappa\sqrt{x^2+s^2}}{2\varepsilon}\right) ds \right].$$

Differentiating, on Π_x we have

$$(z_\mu)_y(x,y) = \frac{x}{2\pi\varepsilon}\, e^{px/(2\varepsilon)} \left[-\frac{\hat{b}_\mu}{2\sqrt{\varepsilon}} \int_{s=-\infty}^{-y} e^{\hat{b}_\mu(s+y)/(2\sqrt{\varepsilon})} \frac{1}{\sqrt{x^2+s^2}} K_1\left(\frac{\kappa\sqrt{x^2+s^2}}{2\varepsilon}\right) ds \right.$$
$$\left. -\frac{\hat{b}_\mu}{2\sqrt{\varepsilon}} \int_{s=-y}^{\infty} e^{-\hat{b}_\mu(s+y)/(2\sqrt{\varepsilon})} \frac{1}{\sqrt{x^2+s^2}} K_1\left(\frac{\kappa\sqrt{x^2+s^2}}{2\varepsilon}\right) ds + \frac{2}{r} K_1\left(\frac{\kappa r}{2\varepsilon}\right) \right],$$

where $r = \sqrt{x^2+y^2}$. Differentiating again, a computation gives

$$(z_\mu)_{yy}(x,y) = \frac{\hat{b}_\mu^2}{4\varepsilon} z_\mu(x,y) + \frac{x}{2\pi\varepsilon}\, e^{px/(2\varepsilon)} \frac{\partial}{\partial y}\left[\frac{2}{r} K_1\left(\frac{\kappa r}{2\varepsilon}\right)\right] \quad \text{on } \Pi_x, \tag{4.7}$$

where we used the earlier formula for $z_\mu(x,y)$. The identity (4.7) gives a simple relationship between z_μ and its even-order y-derivatives. It enables us to bound these derivatives in Lemma 4.5.

Set $\lambda = \kappa r/(2\varepsilon)$ and define the operator σ by

$$\sigma(\cdot) = \frac{1}{\lambda}\frac{\partial(\cdot)}{\partial\lambda}.$$

Now

$$\frac{\partial(\cdot)}{\partial y} = \frac{\partial(\cdot)}{\partial\lambda}\cdot\frac{\partial\lambda}{\partial y} = \frac{\kappa y}{2\varepsilon r}\frac{\partial(\cdot)}{\partial\lambda} = \left(\frac{\kappa^2 y}{4\varepsilon^2}\right)\left(\frac{1}{\lambda}\right)\frac{\partial(\cdot)}{\partial\lambda} = \left(\frac{\kappa^2 y}{4\varepsilon^2}\right)\sigma(\cdot). \tag{4.8}$$

Hence (4.7) can be written as

$$(z_\mu)_{yy}(x,y) = \frac{\hat{b}_\mu^2}{4\varepsilon} z_\mu(x,y) + \frac{\kappa x}{2\pi\varepsilon^2}\, e^{px/(2\varepsilon)} \frac{\kappa^2 y}{4\varepsilon^2}\sigma\left(\frac{1}{\lambda}K_1(\lambda)\right). \tag{4.9}$$

We shall differentiate (4.9) repeatedly to obtain a formula for $D_y^{2k} z_\mu(x,y)$.

Lemma 4.3. *Let k be a non-negative integer. There exist constants $\alpha_{k,m}$, which depend only on k and m, such that*

$$D_y^{2k}\left[\frac{\kappa^2 y}{4\varepsilon^2}\sigma\left(\frac{1}{\lambda}K_1(\lambda)\right)\right] = \sum_{m=k+1}^{2k+1} \alpha_{k,m}\left(\frac{\kappa^2}{4\varepsilon^2}\right)^m y^{2m-2k-1}\sigma^m\left(\frac{1}{\lambda}K_1(\lambda)\right) \tag{4.10}$$

for all $(x,y) \in \Pi_x$.

Proof. Use induction on k. The case $k = 0$ clearly holds true with $\alpha_{0,1} = 1$. Fix $k \ge 0$ and assume that (4.10) is valid for that k and some constants $\alpha_{k,k+1}, \ldots, \alpha_{k,2k+1}$. Define $\alpha_{k,m}$ to be 0 for $m < k+1$ and

$m > 2k+1$; these supernumerary terms enable us to write the sums below in a compact form. Then, using (4.8),

$$D_y^{2k+1}\left[\frac{\kappa^2 y}{4\varepsilon^2}\sigma\left(\frac{1}{\lambda}K_1(\lambda)\right)\right] = \frac{\partial}{\partial y}\left[\sum_{m=k+1}^{2k+1}\alpha_{k,m}\left(\frac{\kappa^2}{4\varepsilon^2}\right)^m y^{2m-2k-1}\sigma^m\left(\frac{1}{\lambda}K_1(\lambda)\right)\right]$$
$$= \sum_{m=k+1}^{2k+2}\Big[(2m-2k-1)\alpha_{k,m}+\alpha_{k,m-1}\Big]\left(\frac{\kappa^2}{4\varepsilon^2}\right)^m y^{2m-2k-2}\sigma^m\left(\frac{1}{\lambda}K_1(\lambda)\right).$$

Differentiating again, a computation gives

$$D_y^{2k+2}\left[\frac{\kappa^2 y}{4\varepsilon^2}\sigma\left(\frac{1}{\lambda}K_1(\lambda)\right)\right]$$
$$= \frac{\partial}{\partial y}\left\{\sum_{m=k+1}^{2k+2}\Big[(2m-2k-1)\alpha_{k,m}+\alpha_{k,m-1}\Big]\left(\frac{\kappa^2}{4\varepsilon^2}\right)^m y^{2m-2k-2}\sigma^m\left(\frac{1}{\lambda}K_1(\lambda)\right)\right\}$$
$$= \sum_{m=k+1}^{2k+2}\Big\{\alpha_{k,m-1}+(4m-4k-1)\alpha_{k,m}+(2m-2k)(2m-2k+1)\alpha_{k,m+1}\Big\}$$
$$\left(\frac{\kappa^2}{4\varepsilon^2}\right)^{m+1} y^{2m-2k-1}\sigma^{m+1}\left(\frac{1}{\lambda}K_1(\lambda)\right)$$
$$= \sum_{m=k+2}^{2k+3}\Big\{\alpha_{k,m-2}+(4m-4k-5)\alpha_{k,m-1}+(2m-2k-2)(2m-2k-1)\alpha_{k,m}\Big\}$$
$$\left(\frac{\kappa^2}{4\varepsilon^2}\right)^{m} y^{2m-2k-3}\sigma^{m}\left(\frac{1}{\lambda}K_1(\lambda)\right),$$

since $2m-2k-2=0$ when $m=k+1$ and $a_{k,k-1}=a_{k,k}=0$. That is, we have shown that (4.10) holds true with k replaced by $k+1$ and $\alpha_{k+1,m}=\alpha_{k,m-2}+(4m-4k-5)\alpha_{k,m-1}+(2m-2k-2)(2m-2k-1)\alpha_{k,m}$ for $m=k+2,\ldots,2k+3$. By the principle of induction, the proof is complete. ∎

We next require an identity.

Lemma 4.4. *For $k=0,\cdots$, one has $D_y^{2k}z(x,y)=A+B$, where*

$$A = \sum_{\mu=0}^{\nu+1} d_\mu\left(\frac{\hat{b}_m^2}{4\varepsilon}\right)^k z_\mu(x,y),$$
$$B = \frac{x}{\pi\varepsilon}e^{px/(2\varepsilon)}\sum_{j=\nu+1}^{k-1}\left[\sum_{\mu=0}^{\nu+1} d_\mu\left(\frac{\hat{b}_m^2}{4\varepsilon}\right)^j\right]\sum_{m=k-j}^{2k-2j-1}(-1)^m\alpha_{k-j-1,m}\left(\frac{\kappa}{2\varepsilon}\right)^m y^{2m-2(k-j)+1}r^{-m-1}K_{m+1}(\lambda),$$

and the outer sum in B is interpreted as 0 if $\nu+1>k-1$.

Proof. If $k=0$ then $B=0$ and the result is obvious. Assume k is a positive integer. Applying D_y^{2k-2} to (4.9) and invoking Lemma 4.3 yields

$$D_y^{2k}z_\mu(x,y) = \frac{\hat{b}_\mu^2}{4\varepsilon}D_y^{2k-2}z_\mu(x,y)+\frac{\kappa x}{2\pi\varepsilon^2}e^{px/(2\varepsilon)}\sum_{m=k}^{2k-1}\alpha_{k-1,m}\left(\frac{\kappa^2}{4\varepsilon^2}\right)^m y^{2m-2k+1}\sigma^m\left(\frac{1}{\lambda}K_1(\lambda)\right).$$

Using this identity recursively, we get

$$\begin{aligned}
&D_y^{2k} z_\mu(x,y) \\
&= \left(\frac{\hat b_\mu^2}{4\varepsilon}\right)^2 D_y^{2k-4} z_\mu(x,y) + \frac{\kappa x}{2\pi\varepsilon^2}\, e^{px/(2\varepsilon)} \left[\frac{\hat b_\mu^2}{4\varepsilon} \sum_{m=k-1}^{2k-3} \alpha_{k-2,m} \left(\frac{\kappa^2}{4\varepsilon^2}\right)^m y^{2m-2k+3} \sigma^m \left(\frac{1}{\lambda} K_1(\lambda)\right)\right. \\
&\qquad\qquad \left. + \sum_{m=k}^{2k-1} \alpha_{k-1,m} \left(\frac{\kappa^2}{4\varepsilon^2}\right)^m y^{2m-2k+1} \sigma^m \left(\frac{1}{\lambda} K_1(\lambda)\right)\right] \\
&= \cdots \\
&= \left(\frac{\hat b_\mu^2}{4\varepsilon}\right)^k z_\mu(x,y) + \frac{\kappa x}{2\pi\varepsilon^2}\, e^{px/(2\varepsilon)} \sum_{j=0}^{k-1} \left(\frac{\hat b_\mu^2}{4\varepsilon}\right)^j \sum_{m=k-j}^{2k-2j-1} \alpha_{k-j-1,m} \left(\frac{\kappa^2}{4\varepsilon^2}\right)^m y^{2m-2(k-j)+1} \sigma^m \left(\frac{1}{\lambda} K_1(\lambda)\right).
\end{aligned}$$

Hence $D_y^{2k} z(x,y) = A + B_1$ where, using the compatibility condition (4.1),

$$\begin{aligned}
B_1 &= \frac{\kappa x}{2\pi\varepsilon^2}\, e^{px/(2\varepsilon)} \sum_{j=0}^{k-1} \left[\sum_{\mu=0}^{\nu+1} d_\mu \left(\frac{\hat b_\mu^2}{4\varepsilon}\right)^j\right] \sum_{m=k-j}^{2k-2j-1} \alpha_{k-j-1,m} \left(\frac{\kappa^2}{4\varepsilon^2}\right)^m y^{2m-2(k-j)+1} \sigma^m \left(\frac{1}{\lambda} K_1(\lambda)\right) \\
&= \frac{\kappa x}{2\pi\varepsilon^2}\, e^{px/(2\varepsilon)} \sum_{j=\nu+1}^{k-1} \left[\sum_{\mu=0}^{\nu+1} d_\mu \left(\frac{\hat b_\mu^2}{4\varepsilon}\right)^j\right] \sum_{m=k-j}^{2k-2j-1} \alpha_{k-j-1,m} \left(\frac{\kappa^2}{4\varepsilon^2}\right)^m y^{2m-2(k-j)+1} \sigma^m \left(\frac{1}{\lambda} K_1(\lambda)\right),
\end{aligned}$$

by (4.1). Now [1, (9.6.28)] provides the useful identity

$$\sigma^m \left(\frac{1}{\lambda} K_1(\lambda)\right) = (-1)^m \lambda^{-m-1} K_{m+1}(\lambda),$$

so

$$B_1 = \frac{\kappa x}{2\pi\varepsilon^2}\, e^{px/(2\varepsilon)} \sum_{j=\nu+1}^{k-1} \left[\sum_{\mu=0}^{\nu+1} d_\mu \left(\frac{\hat b_\mu^2}{4\varepsilon}\right)^j\right] \sum_{m=k-j}^{2k-2j-1} (-1)^m \alpha_{k-j-1,m} \left(\frac{\kappa^2}{4\varepsilon^2}\right)^m y^{2m-2(k-j)+1} \lambda^{-m-1} K_{m+1}(\lambda).$$

Recalling that $\lambda = \kappa r/(2\varepsilon)$ gives $B_1 = B$. ■

Now we can proceed with the estimation of derivatives. Set

$$H(t) = \begin{cases} 0 & \text{if } t \le 0, \\ 1 & \text{if } t > 0, \end{cases}$$

and

$$\phi_3(x,y) = \exp\left(-\frac{py^2}{16\varepsilon\sqrt{x^2+y^2}}\right) \exp\left(-\frac{qx}{2p}\right).$$

Lemma 4.5. *Let $r^* \ge \varepsilon$ be given. For $\varepsilon \le r \le r^*$ and $k = 0, 1, \ldots$, there is a $C = C(r^*, k, \nu)$ such that*

$$\left|D_y^{2k} z(x,y)\right| \le C\varepsilon^{-k} \left[\phi_2(x,|y|) + H(k-\nu-1) r^{\nu+1-k} \phi_3(x,y)\right]. \tag{4.11}$$

Proof. If $k = 0$ the result is immediate from Lemma 4.2, so assume that k is a positive integer. Let $(x,y) \in \Pi_x$ with $\varepsilon \le r \le r^*$. Lemma 4.2 implies that

$$A \le C\varepsilon^{-k} \sum_{\mu=0}^{\nu+1} |d_\mu|\ |z_\mu(x,y)| \le C\varepsilon^{-k} \phi_2(x,|y|). \tag{4.12}$$

For $r \geq \varepsilon$ we have $\lambda \geq \kappa/2 \geq p/2 > 0$, so by [1, (9.7.2)] there exists C such that

$$0 < K_i(\lambda) \leq C\lambda^{-1/2}e^{-\lambda} \text{ for } 0 \leq i \leq 2k+1. \tag{4.13}$$

Thus in estimating B one must bound $\exp[(px - \kappa r)/(2\varepsilon)]$. Now

$$\frac{px - \kappa r}{2\varepsilon} = \frac{p^2x^2 - \kappa^2 r^2}{2\varepsilon(px + \kappa r)} = \frac{-p^2y^2 - 4\varepsilon q r^2}{2\varepsilon(px + \kappa r)} \leq \frac{-p^2y^2 - 4\varepsilon q r^2}{4\varepsilon\kappa r} \leq -\frac{py^2}{8\varepsilon r} - \frac{qx}{2p}, \tag{4.14}$$

where in the final inequality we used $\kappa = \sqrt{p^2 + 4\varepsilon q} \leq 2p$ for ε sufficiently small. Applying these inequalities and using $x \leq r$, the quantity B is bounded by

$$\begin{aligned}
C\varepsilon^{-1} \sum_{j=\nu+1}^{k-1} & \varepsilon^{-j} \sum_{m=k-j}^{2k-2j-1} \varepsilon^{-m}|y|^{2m-2(k-j)+1} r^{-m} \left(\frac{\varepsilon}{r}\right)^{1/2} \exp\left(-\frac{py^2}{8\varepsilon r}\right) \exp\left(-\frac{qx}{2p}\right) \\
&= C \sum_{j=\nu+1}^{k-1} \sum_{m=k-j}^{2k-2j-1} \varepsilon^{-k} r^{j-k} \left(\frac{y^2}{\varepsilon r}\right)^{m-(k-j)+1/2} \exp\left(-\frac{py^2}{8\varepsilon r}\right) \exp\left(-\frac{qx}{2p}\right) \\
&\leq C \sum_{j=\nu+1}^{k-1} \sum_{m=k-j}^{2k-2j-1} \varepsilon^{-k} r^{j-k} \exp\left(-\frac{py^2}{16\varepsilon r}\right) \exp\left(-\frac{qx}{2p}\right) \\
&\leq C\varepsilon^{-k} r^{\nu+1-k} \phi_3(x,y),
\end{aligned} \tag{4.15}$$

since $r \leq r^*$, where we used the standard inequality $t^\lambda e^{-Ct} \leq C$ for all $t \geq 0$ and fixed $\lambda > 0$. Using our bound (4.12) for A, using (4.15) to bound B, and recalling that B vanishes when $\nu + 1 > k - 1$, the proof is complete. ∎

In the next result the assertion of Theorem 4.5 is simplified by absorbing ϕ_3 into ϕ_2 and by removing the function H.

Corollary 4.6. *Let $r^* \geq \varepsilon$ be given. For $\varepsilon \leq r \leq r^*$ and $k = 0, 1, \ldots,$ there is a $C = C(r^*, k, \nu)$ such that*

$$\left|D_y^{2k} z(x,y)\right| \leq C\left[\varepsilon^{-k} + \varepsilon^{-k} r^{\nu+1-k}\right] \phi_2(x,|y|). \tag{4.16}$$

Proof. We assert that there is a constant C such that $\phi_3(x,y) \leq C\phi_2(x,|y|)$ on Π_x. If $py^2/(16\varepsilon r) > \beta|y|/(2\sqrt{\varepsilon})$, then clearly $\phi_3(x,y) \leq \phi_2(x,|y|)$. If $py^2/(16\varepsilon r) \leq \beta|y|/(2\sqrt{\varepsilon})$, then $|y| \leq 8\beta\sqrt{\varepsilon}\, r/p$, so

$$\exp\left(\frac{4\beta^2 r^*}{p}\right) \phi_2(x,|y|) \geq \exp\left(\frac{-qx}{2p}\right) \exp\left(\frac{4\beta^2 r^* - 4\beta^2 r}{p}\right) \geq \exp\left(\frac{-qx}{2p}\right) \geq \phi_3(x,y),$$

so the assertion is shown. Combining this fact with $0 \leq H(\cdot) \leq 1$, (4.16) follows from Lemma 4.5. ∎

We now treat the case $r \leq \varepsilon$.

Lemma 4.7. *For $k = 0, 1, \ldots,$ there is a $C = C(k, \nu)$ such that if $r \leq 2\varepsilon$,*

$$\left|D_y^{2k} z(x,y)\right| \leq C\left[\varepsilon^{-k} + \varepsilon^{-\nu-1} r^{2\nu+2-2k}\right]. \tag{4.17}$$

Proof. Taking $r \leq \varepsilon$ and referring to Lemma 4.4 one has $|A| \leq C\varepsilon^{-k}$. Since $|K_{m+1}(\lambda)| \leq C'\lambda^{-m-1}$ for

$\lambda \le C$ (see [1, (9.6.9)]),

$$
\begin{aligned}
|B| &\le Cx\varepsilon^{-1} \sum_{j=\nu+1}^{k-1} \left[\sum_{\mu=0}^{\nu+1} \varepsilon^{-j}\right] \sum_{m=k-j}^{2k-2j-1} \varepsilon^{-m} y^{2m-2(k-j)+1} r^{-m-1} K_{m+1}(\lambda) \\
&\le Cx\varepsilon^{-1} \sum_{j=\nu+1}^{k-1} \varepsilon^{-j} \sum_{m=k-j}^{2k-2j-1} \varepsilon^{-m} y^{2m-2(k-j)+1} r^{-m-1} (r/\varepsilon)^{-m-1} \\
&= Cxr^{-2}\varepsilon^{-1} \sum_{j=\nu+1}^{k-1} \varepsilon^{1-j} y^{-2(k-j)+1} \sum_{m=k-j}^{2k-2j-1} y^{2m} r^{-2m} \\
&\le Cxr^{-2}\varepsilon^{-1} \sum_{j=\nu+1}^{k-1} \varepsilon^{1-j} y^{-2(k-j)+1} y^{2(k-j)} r^{-2(k-j)} \\
&= Cxyr^{-2-2k} \sum_{j=\nu+1}^{k-1} \varepsilon^{-j} r^{2j} \\
&\le Cxyr^{-2-2k} \varepsilon^{-\nu-1} r^{2\nu+2} \\
&= C\varepsilon^{-\nu-1} r^{2\nu+2-2k}.
\end{aligned}
$$

In the penultimate step we have used the fact that $0 < r^2/\varepsilon \le 4\varepsilon \le 4$. We therefore get

$$
|D_y^{2k} z(x,y)| \le C(\varepsilon^{-k} + \varepsilon^{-\nu-1} r^{2\nu+2-2k}) \text{ for } r \le \varepsilon.
$$

■

To get estimates for odd order derivatives we need an interpolation inequality.

Lemma 4.8. *Let $f \in C^2[y, y+\delta]$ for some $\delta > 0$ and some y. Then*

$$
|f'(y)| \le 2\delta^{-1} \max_{\eta \in [y,y+\delta]} |f(\eta)| + \delta \max_{\eta \in [y,y+\delta]} |f''(\eta)|.
$$

Proof. By the mean value theorem there exists $\eta' \in (y, y+\delta)$ such that

$$
|f'(\eta')| = \delta^{-1} |f(y+\delta) - f(y)| \le 2\delta^{-1} \max_{\eta \in [y,y+\delta]} |f(\eta)|.
$$

Hence

$$
|f'(y)| = \left| f'(\eta') - \int_y^{\eta'} f''(s)\, ds \right| \le 2\delta^{-1} \max_{\eta \in [y,y+\delta]} |f(y)| + \delta \max_{\eta \in [y,y+\delta]} |f''(y)|.
$$

■

The odd-order y-derivatives of z can now be estimated.

Lemma 4.9. *Let $r^* \ge \varepsilon$ be given. For $k = 0, 1, \ldots$, there is a $C = C(r^*, k, \nu)$ such that*

(4.18a) $$|D_y^{2k+1} z(x,y)| \le C[\varepsilon^{-k-1/2} + \varepsilon^{-\nu-1} r^{2\nu+1-2k}] \text{ for } r \le \varepsilon,$$

(4.18b) $$|D_y^{2k+1} z(x,y)| \le C[\varepsilon^{-k-1/2} + \varepsilon^{-k-1/2} r^{\nu-k+1/2}] \phi_2(x, |y|) \text{ for } \varepsilon \le r \le r^*.$$

Proof. As $z(x,y)$ is an odd function of y, without loss of generality we can take $y \ge 0$. Let (x,y) be a point with $r \le \varepsilon$. Set $I = [y, y+r]$ and $r' = \sqrt{x^2 + y'^2}$ where $y' \in I$. The quantities r/r' and r'/r are bounded in I. From Lemma 4.7,

$$
\max_{y' \in I} |D_y^{2k} z(x,y')| \le C(\varepsilon^{-k} + \varepsilon^{-\nu-1} r^{2\nu-2k+2}), \quad \max_{y' \in I} |D_y^{2k+2} z(x,y')| \le C(\varepsilon^{-k-1} + \varepsilon^{-\nu-1} r^{2\nu-2k}).
$$

Using Lemma 4.8 with $f(y) = D_y^{2k} z(x,y)$ and $\delta = r$ we obtain

$$|D_y^{2k+1} z(x,y)| \le C(r^{-1}\varepsilon^{-k} + \varepsilon^{-\nu-1} r^{2\nu-2k+1} + r\varepsilon^{-k-1} + \varepsilon^{-\nu-1} r^{2\nu-2k+1}).$$

We want to show that each of the 4 terms on the right hand side of this inequality is bounded by the right hand side of (4.18a). For the second and fourth terms this is seen by inspection. For the first term, we have $r^{-1}\varepsilon^{-k} \le C\varepsilon^{-\nu-1} r^{2\nu-2k+1}$ provided $r^{2k-2\nu-2} \le C\varepsilon^{k-\nu-1}$, and this is true since $r^2 \le r \le \varepsilon$. For the third term, since $r \le C\varepsilon$ we have $r\varepsilon^{-k-1} \le \varepsilon^{-k-1/2}$, finishing the proof of (4.18a).

Next, let (x,y) be a point with $\varepsilon \le r \le r^*$. Set $J = [y, y + \varepsilon^{1/2} r]$ and $r' = \sqrt{x^2 + y'^2}$ where $y' \in J$. The quantities r/r' and r'/r are bounded for $y' \in J$. Also, $\phi_2(x, |y'|) \le \phi_2(x, |y|)$ for $y' \in J$. Therefore, from Corollary 4.6,

(4.19)

$$\max_{y' \in J} |D_y^{2k} z(x,y')| \le C(\varepsilon^{-k} + \varepsilon^{-k} r^{\nu-k+1})\phi_2(x,|y|),\ \max_{y' \in J} |D_y^{2k+2} z(x,y')| \le C(\varepsilon^{-k-1} + \varepsilon^{-k-1} r^{\nu-k})\phi_2(x,|y|).$$

Suppose $\nu < k$. By Lemma 4.8 with $f(y) = D_y^{2k} z(x,y)$ and $\delta = \varepsilon^{1/2} r^{1/2}$, one gets

$$|D_y^{2k+1} z(x,y)| \le C(\varepsilon^{-k-1/2} r^{-1/2} + \varepsilon^{-k-1/2} r^{\nu-k+1/2} + \varepsilon^{-k-1/2} r^{1/2} + \varepsilon^{-k-1/2} r^{\nu-k+1/2})\phi_2(x,|y|).$$

We want to show that each of the 4 terms on the right hand side of this inequality is bounded by the right hand side of (4.18b). For the second and fourth terms this is seen by inspection. Since $\nu \le k-1$ we have $r^{\pm 1/2} \le r^{\nu-k+1/2}$ for $r \le r^*$. Applying this inequality to the first and third terms, we obtain (4.18b). In the case $\nu \ge k$, inequality (4.19) implies

$$\max_{y' \in I} |D_y^{2k} z(x,y')| \le C\varepsilon^{-k}\phi_2(x,|y|),\ \max_{y' \in I} |D_y^{2k+2} z(x,y')| \le C\varepsilon^{-k-1}\phi_2(x,|y|) \text{ for } \varepsilon \le r \le r^*.$$

Lemma 4.9 with $\delta = \varepsilon^{1/2}$ then gives (4.18b). ■

The main result of this section follows. It extends the previous bounds to all derivatives of z.

Theorem 4.10. *Let $r^* \ge \varepsilon$ be given. Let m and n be non-negative integers. Then there exist a constant C, which depends on r^*, m, n and ν, such that*

$$\left|D_x^m D_y^n z(x,y)\right| \le C\left[\varepsilon^{-n/2} + \varepsilon^{\nu+1-m-n} + \varepsilon^{-\nu-1}\psi(\nu,m,n,r)\right] \text{ for } r < \varepsilon, \tag{4.20a}$$

$$\left|D_x^m D_y^n z(x,y)\right| \le C\varepsilon^{-n/2}\left[1 + r^{\nu+1-m-n/2}\right]\phi_2(x,|y|) \text{ for } \varepsilon \le r \le r^*, \tag{4.20b}$$

where

$$\psi(\nu,m,n,r) = \begin{cases} r^{2\nu+2-m-n}|\ln r| & \text{if } m+n \le 2\nu+2, \\ r^{2\nu+2-m-n} & \text{if } m+n > 2\nu+2. \end{cases}$$

Proof. We use induction on m. The case $m = 0$ is covered by Corollary 4.6, Lemma 4.7, and Lemma 4.9. Let M be a non-negative integer. Assume that (4.20) holds true for $m = M$ and all $n \ge 0$. We shall use a variant of the argument in [6, Lemma 2.2] to deduce a bound on $|D_x^{M+1} D_y^n z(x,y)|$, where $n \ge 0$ is fixed and satisfies $(M+1) + n > 2\nu + 1$.

Now $0 = (D_x^M D_y^n)Lz = L(D_x^M D_y^n z)$ on Π_x. That is,

$$(-\varepsilon w_{xx} + p w_x)(x,y) = s(x,y), \tag{4.21}$$

where we set $w(x,y) = D_x^M D_y^n z(x,y)$ and $s(x,y) = (\varepsilon D_x^M D_y^{n+2} z + q D_x^M D_y^n z)(x,y)$.

Fix $(x, y) \in \Pi_x$ with $r \leq 1$. Let $\xi \in (x, x+1)$. Multiplying (4.21) by the integrating factor $-\varepsilon^{-1}e^{-px/\varepsilon}$ then integrating from ξ to $x+1$, we obtain

$$w_x(x+1, y)e^{-p(x+1)/\varepsilon} - w_x(\xi, y)e^{-p\xi/\varepsilon} = -\varepsilon^{-1}\int_{t=\xi}^{x+1} e^{-pt/\varepsilon}s(t, y)\,dt,$$

i.e.,

$$w_x(\xi, y) = w_x(x+1, y)e^{-p(x+1-\xi)/\varepsilon} + \varepsilon^{-1}\int_{t=\xi}^{x+1} e^{-p(t-\xi)/\varepsilon}s(t, y)\,dt. \tag{4.22}$$

Integrate (4.22) from $\xi = x$ to $\xi = x+1$:

$$w(x+1, y) - w(x, y) = \varepsilon p^{-1}(1 - e^{-p/\varepsilon})w_x(x+1, y) + \varepsilon^{-1}\int_{\xi=x}^{x+1}\int_{t=\xi}^{x+1} e^{-p(t-\xi)/\varepsilon}s(t, y)\,dt\,d\xi.$$

Hence

$$w_x(x+1, y) = \frac{p}{\varepsilon(1 - e^{-p/\varepsilon})}\left[w(x+1, y) - w(x, y) - \varepsilon^{-1}\int_{\xi=x}^{x+1}\int_{t=\xi}^{x+1} e^{-p(t-\xi)/\varepsilon}s(t, y)\,dt\,d\xi\right]. \tag{4.23}$$

In (4.22) take $\xi = x$ and substitute (4.23):

$$\begin{aligned} w_x(x, y) &= \frac{pe^{-p/\varepsilon}}{\varepsilon(1 - e^{-p/\varepsilon})}\left[w(x+1, y) - w(x, y) - \varepsilon^{-1}\int_{\xi=x}^{x+1}\int_{t=\xi}^{x+1} e^{-p(t-\xi)/\varepsilon}s(t, y)\,dt\,d\xi\right] \\ &\qquad + \varepsilon^{-1}\int_{t=x}^{x+1} e^{-p(t-x)/\varepsilon}s(t, y)\,dt \\ &= \frac{pe^{-p/\varepsilon}}{\varepsilon(1 - e^{-p/\varepsilon})}\left[w(x+1, y) - w(x, y) - \int_{t=x}^{x+1} s(t, y)\int_{\xi=x}^{t} \varepsilon^{-1}e^{-p(t-\xi)/\varepsilon}\,d\xi\,dt\right] \\ &\qquad + \varepsilon^{-1}\int_{t=x}^{x+1} e^{-p(t-x)/\varepsilon}s(t, y)\,dt. \end{aligned} \tag{4.24}$$

But $\int_{\xi=x}^{t} \varepsilon^{-1}e^{-p(t-\xi)/\varepsilon}\,d\xi \leq C$ and $\varepsilon^{-1}\int_{t=x}^{x+1} e^{-p(t-x)/\varepsilon}\,dt \leq C$, so the identity (4.24) implies that

$$|w_x(x, y)| \leq C\Big[|w(x+1, y)| + |w(x, y)| + \max_{x \leq t \leq x+1}|s(t, y)|\Big]. \tag{4.25}$$

Suppose that $\varepsilon \leq r \leq r^*$. For fixed y (and fixed m, n), the bound (4.20b) is monotonically decreasing as a function of x if $m + n/2 \geq \nu + 1$, while if $m + n/2 < \nu + 1$ then $r^{\nu+1-m-n/2} \leq C$. In both cases it follows that when using (4.20b) to bound terms of the form $|w(\cdot, y)|$ and $|s(\cdot, y)|$, the worst case occurs when the first argument is as small as possible. Thus (4.25) and the inductive hypothesis imply that

$$\begin{aligned} |w_x(x, y)| &\leq C\varepsilon^{-n/2}\left[1 + r^{\nu+1-M-n/2}\right]\phi_2(x, |y|) + C\varepsilon\,\varepsilon^{-(n+2)/2}\left[1 + r^{\nu+1-M-(n+2)/2}\right]\phi_2(x, |y|) \\ &\leq C\varepsilon^{-n/2}\left[1 + r^{\nu+1-(M+1)-n/2}\right]\phi_2(x, |y|), \end{aligned} \tag{4.26}$$

which is the desired inequality.

Now suppose that $r < \varepsilon$. Instead of using (4.24), we multiply (4.21) by the integrating factor $-\varepsilon^{-1}e^{-px/\varepsilon}$ then integrate from x to $\sqrt{\varepsilon^2 - y^2}$. This yields

$$w_x(\sqrt{\varepsilon^2 - y^2}, y)e^{-p\sqrt{\varepsilon^2 - y^2}/\varepsilon} - w_x(x, y)e^{-px/\varepsilon} = -\varepsilon^{-1}\int_{t=x}^{\sqrt{\varepsilon^2 - y^2}} e^{-pt/\varepsilon}s(t, y)\,dt,$$

i.e.,

$$w_x(x,y) = w_x(\sqrt{\varepsilon^2 - y^2}, y)e^{-p(\sqrt{\varepsilon^2-y^2}-x)/\varepsilon} + \varepsilon^{-1}\int_{t=x}^{\sqrt{\varepsilon^2-y^2}} e^{-p(t-x)/\varepsilon} s(t,y)\,dt.$$

Hence

$$|w_x(x,y)| \le |w_x(\sqrt{\varepsilon^2 - y^2}, y)| + \varepsilon^{-1}\int_{t=x}^{\sqrt{\varepsilon^2-y^2}} e^{-p(t-x)/\varepsilon} |s(t,y)|\,dt. \tag{4.27}$$

We use (4.26) to bound $|w_x(\sqrt{\varepsilon^2 - y^2}, y)|$ and the inductive hypothesis to bound $|s(t,y)|$, obtaining

$$\begin{aligned}
|w_x(x,y)| \le C\Big\{&\varepsilon^{-n/2}\left[1 + \varepsilon^{\nu+1-(M+1)-n/2}\right] \\
&+ \varepsilon^{-1}\int_{t=x}^{\sqrt{\varepsilon^2-y^2}} e^{-p(t-x)/\varepsilon}\left[\varepsilon^{-n/2} + \varepsilon^{\nu+1-M-n} + \varepsilon^{-\nu-1}\psi(\nu, M, n, \sqrt{t^2+y^2})\right] dt \\
&+ \int_{t=x}^{\sqrt{\varepsilon^2-y^2}} e^{-p(t-x)/\varepsilon}\left[\varepsilon^{-(n+2)/2} + \varepsilon^{\nu+1-M-(n+2)} + \varepsilon^{-\nu-1}\psi(\nu, M, n+2, \sqrt{t^2+y^2})\right] dt\Big\} \\
\le C&\left\{\varepsilon^{-n/2} + \varepsilon^{\nu-M-n} + \varepsilon^{-\nu-1}\int_{t=x}^{\sqrt{\varepsilon^2-y^2}} \psi(\nu, M, n+2, \sqrt{t^2+y^2})\,dt\right\} \\
\le C&\left\{\varepsilon^{-n/2} + \varepsilon^{\nu-M-n} + \varepsilon^{-\nu-1}\int_{t=x}^{\sqrt{\varepsilon^2-y^2}} \psi(\nu, M, n+2, t+y)\,dt\right\} \\
\le C&\left\{\varepsilon^{-n/2} + \varepsilon^{\nu+1-(M+1)-n} + \varepsilon^{-\nu-1}\psi(\nu, M+1, n, r)\right\}
\end{aligned} \tag{4.28}$$

on evaluating the integral. This is the desired inequality.

This completes the inductive step, and the theorem is proved. ■

If $m + n \ge 2\nu + 2$ and $r < \varepsilon$ then $\varepsilon^{\nu+1-m-n} \le C\varepsilon^{-\nu-1}\psi(\nu, m, n, r)$. Also, $\psi \le C$ if $m + n < 2\nu + 2$. It follows that (4.20a) implies

$$\left|D_x^m D_y^n z(x,y)\right| \le C\left[\varepsilon^{-n/2} + \varepsilon^{\nu+1-m-n}\right] \text{ for } m+n < 2\nu+2 \text{ and } r < \varepsilon, \tag{4.29a}$$

$$\left|D_x^m D_y^n z(x,y)\right| \le C\left[\varepsilon^{-n/2} + \varepsilon^{-\nu-1}|\ln r|\right] \text{ for } m+n = 2\nu+2 \text{ and } r < \varepsilon, \tag{4.29b}$$

$$\left|D_x^m D_y^n z(x,y)\right| \le C\left[\varepsilon^{-n/2} + \varepsilon^{-\nu-1}r^{2\nu+2-m-n}\right] \text{ for } m+n > 2\nu+2 \text{ and } r < \varepsilon. \tag{4.29c}$$

Using Theorem 4.10 we return to the quarter-plane problem of Section 2 and prove

Theorem 4.11. *Let g, h satisfy (2.2) and suppose the first $\nu+1$ compatibility conditions for the problem (2.1) are satisfied. Let u satisfy (2.1). Then for $r^* \ge \varepsilon$ and integers m and n satisfying $2m + n \le 2\ell$ there is a constant C depending on r^*, ℓ and ν such that*

$$\left|D_x^m D_y^n u(x,y)\right| \le C(\bar{G}_{2\ell} + \bar{H}_{2\ell})\left[\varepsilon^{-n/2} + \varepsilon^{\nu+1-m-n}\right] \text{ for } m+n < 2\nu+2 \text{ and } r < \varepsilon, \tag{4.30a}$$

$$\left|D_x^m D_y^n u(x,y)\right| \le C(\bar{G}_{2\ell} + \bar{H}_{2\ell})\left[\varepsilon^{-n/2} + \varepsilon^{-\nu-1}|\ln r|\right] \text{ for } m+n = 2\nu+2 \text{ and } r < \varepsilon, \tag{4.30b}$$

$$\left|D_x^m D_y^n u(x,y)\right| \le C(\bar{G}_{2\ell} + \bar{H}_{2\ell})\left[\varepsilon^{-n/2} + \varepsilon^{-\nu-1}r^{2\nu+2-m-n}\right] \text{ for } m+n > 2\nu+2 \text{ and } r < \varepsilon. \tag{4.30c}$$

$$\left|D_x^m D_y^n u(x,y)\right| \le C(\bar{G}_{2\ell} + \bar{H}_{2\ell})\varepsilon^{-n/2}\left[1 + r^{\nu+1-m-n/2}\right] e^{-qx/(2p)}e^{-\beta y/(2\sqrt{\varepsilon})} \text{ for } \varepsilon \le r \le r^*. \tag{4.30d}$$

Proof. We use the decomposition (2.15). From Theorem 3.10, $|D_x^m D_y^n u_1| \le C\varepsilon^{-n/2}e^{-qx/(2p)}e^{-\sqrt{q}y/(2\sqrt{\varepsilon})}$ for $m + n \le m + \bar{n} \le 2\ell$. Setting $m = 0$ it follows that (2.6) holds, and therefore (2.14) holds. Applying

Theorem 3.6 we find that $|D_x^m D_y^n u_4| \le C\varepsilon^{-n/2} e^{-qx/(2p)} e^{-\sqrt{q}y/(2\sqrt{\varepsilon})}$ for $2m+n \le 2\ell$. The derivatives of z are bounded using Theorem 4.10. ■

5. The unit square problem

We now return to the problem (1.1). Our purpose is to derive pointwise bounds for the derivatives of the solution u. Our method consists in expressing the solution as a sum of half-plane problems and quarter-plane problems (see (5.13)) plus a remainder term. The remainder term satisfies (1.1) with data that is both completely compatible at the corners and exponentially small. The final bounds show the various phenomena experienced by u: boundary layer on the side $x=1$, characteristic boundary layers on the sides $y=0,1$, and corner singularities at the four corners of Q.

To start, let f^* be a smooth extension of f from Q to the half plane $x>0$. Also, let g_w^* and g_e^* be smooth extensions of g_w and g_e from $[0,1]$ to $(-\infty,\infty)$, and let g_s^* and g_n^* be smooth extensions of g_s and g_n from $[0,1]$ to $[0,\infty)$. Let U be the solution to the *incoming half-plane problem*

$$\begin{aligned} LU &= f^* \text{ for } x>0, \\ U(0,y) &= g_w^*(y) \text{ for } -\infty<y<\infty. \end{aligned} \tag{5.1}$$

Similarly, let W be the solution to the *outgoing half-plane problem*

$$\begin{aligned} LW &= 0 \text{ for } x<1, \\ W(1,y) &= g_e^*(y) - U(1,y) \text{ for } -\infty<y<\infty. \end{aligned} \tag{5.2}$$

Sharp derivative bounds for U and W are given in Theorem 3.5 and Theorem 3.14.

Next we define "incoming corner functions" z_{00} and z_{01} as solutions to the following quarter-plane problems:

$$\begin{aligned} Lz_{00} &= 0 \text{ for } 0<x,\ 0<y, \\ z_{00}(x,0) &= g_s^*(x) - U(x,0) \text{ for } 0<x, \\ z_{00}(0,y) &= 0 \text{ for } 0<y; \end{aligned} \tag{5.3}$$

$$\begin{aligned} Lz_{01} &= 0 \text{ for } 0<x, \text{ and } y<1, \\ z_{01}(x,1) &= g_n^*(x) - U(x,1) \text{ for } 0<x, \\ z_{01}(0,y) &= 0 \text{ for } y<1. \end{aligned} \tag{5.4}$$

Set $u^1 = u - U - W - z_{00} - z_{01}$. Then u^1 is the solution of the problem

$$\begin{aligned} Lu^1 &= 0 \text{ in } Q, \\ u^1(x,0) &= g_s^1(x),\ u^1(x,1) = g_n^1(x) \text{ for } 0<x<1, \\ u^1(0,y) &= g_w^1(y),\ u^1(1,y) = g_e^1(y) \text{ for } 0<y<1, \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} g_s^1(x) &= -W(x,0) - z_{01}(x,0), \\ g_n^1(x) &= -W(x,1) - z_{00}(x,1), \\ g_w^1(y) &= -W(0,y), \\ g_e^1(y) &= -z_{00}(1,y) - z_{01}(1,y). \end{aligned} \tag{5.6}$$

Theorem 4.11 gives information regarding the functions z_{00} and z_{01}. The values of z_{00}, z_{01} and g_w^1 that appear in the data (5.6) are exponentially small. The data of the problem (5.5) are compatible to arbitrary order at the corners $(0,0)$ and $(0,1)$, but the compatibility conditions at the corners $(1,0)$ and $(1,1)$ are not necessarily satisfied. To handle these incompatibilities we define functions z_{10} and z_{11} as solutions to the some quarter-plane problems. For this, let $\chi(t)$ be a smooth function on $(0,1)$ with $\chi(t)=0$ near $t=0$ and $\chi(t)=1$ near $t=1$. Define z_{10} and z_{11} by the problems

$$\begin{aligned} Lz_{10} &= 0 \text{ for } x<1,\ 0<y, \\ z_{10}(x,0) &= -\chi(x)W(x,0) \text{ for } x<1, \\ z_{10}(1,y) &= -\chi(1-y)z_{00}(1,y) \text{ for } 0<y; \end{aligned} \tag{5.7}$$

$$\begin{aligned} Lz_{11} &= 0 \text{ for } x<1,\ y<1, \\ z_{11}(x,1) &= -\chi(x)W(x,1) \text{ for } x<1, \\ z_{11}(1,y) &= -\chi(y)z_{01}(1,y) \text{ for } y<1. \end{aligned} \tag{5.8}$$

We will need some information concerning the "outgoing corner functions" z_{10} and z_{11}. This is derived in the following lemma.

Lemma 5.1. *Let $r^* \geq \varepsilon$ be given. Let g and h be functions on $\mathbb{R}_+$ that satisfy*

$$|g^{(j)}(x)| \leq G_j \varepsilon^{-j} e^{-px/\varepsilon} \text{ for } j=0,\cdots 2\ell,$$
$$|h^{(j)}(y)| \leq H_j \varepsilon^{-j/2} e^{-cy/\sqrt{\varepsilon}} \text{ for } j=0,\cdots,2\ell,$$

for suitable sequences $\{G_j\},\{H_j\}$. Let p,q be positive and let z satisfy

$$\begin{aligned} L^*z = -\varepsilon\Delta z - pz_x + qz &= 0 \text{ in } \mathbb{Q}, \\ z(x,0) &= g(x) \text{ for } x>0, \\ z(0,y) &= h(y) \text{ for } y>0. \end{aligned} \tag{5.9}$$

Suppose the first $\nu+1$ compatibility conditions for the problem (5.9) are satisfied. Then for integers m and n satisfying $2m+n\leq 2\ell$ there is a constant C depending on ℓ,ν and r^* such that

$$\left|D_x^m D_y^n z(x,y)\right| \leq C\varepsilon^{-2m-\bar{n}}\left[\varepsilon^{-n/2}+\varepsilon^{\nu+1-m-n}\right] \text{ for } m+n<2\nu+2 \text{ and } r<\varepsilon, \tag{5.10a}$$

$$\left|D_x^m D_y^n z(x,y)\right| \leq C\varepsilon^{-2m-\bar{n}}\left[\varepsilon^{-n/2}+\varepsilon^{-\nu-1}|\ln r|\right] \text{ for } m+n=2\nu+2 \text{ and } r<\varepsilon, \tag{5.10b}$$

$$\left|D_x^m D_y^n z(x,y)\right| \leq C\varepsilon^{-2m-\bar{n}}\left[\varepsilon^{-n/2}+\varepsilon^{-\nu-1}r^{2\nu+2-m-n}\right] \text{ for } m+n>2\nu+2 \text{ and } r<\varepsilon. \tag{5.10c}$$

$$\left|D_x^m D_y^n z(x,y)\right| \leq C\varepsilon^{-2m-\bar{n}}\left[1+r^{\nu+1-m-n/2}\right]e^{-px/\varepsilon-cy/\sqrt{\varepsilon}} \text{ for } \varepsilon\leq r\leq r^*. \tag{5.10d}$$

Proof. We make a transformation of the problem. Let $z=e^{-px/\varepsilon}v$. Then

$$L^*z = L^*(e^{-px/\varepsilon}v) = e^{-px/\varepsilon}\big\{-\varepsilon\Delta v + pv_x + qv\big\},$$

so v satisfies the boundary value problem

$$\begin{aligned} -\varepsilon\Delta v + pv_x + qv &= 0 \text{ in } \mathbb{Q}, \\ v(x,0) &= g_1(x) \text{ for } x>0, \\ v(0,y) &= h(y) \text{ for } y>0, \end{aligned}$$

where $g_1 = e^{px/\varepsilon} g$ satisfies

$$|g_1^{(m)}(x)| \le C \sum_{m_1+m_2=m} |D^{m_1} e^{px/\varepsilon}||g^{(m_2)}(x)| \le C\varepsilon^{-m}.$$

We apply Theorem 4.11 with $\ell = m + \frac{1}{2}\bar{n}$ and $\bar{G}_\ell = \varepsilon^{-\ell} G_\ell$. Since $2m + n \le 2\ell$, Theorem 4.11 and the chain rule give the asserted estimates for $D_x^m D_y^n z$. ■

The derivative bounds in Lemma 5.1 contain both corner singularities and a rapid exponential decay away from $x = 1$.

Setting $u^2 = u^1 - z_{10} - z_{11}$, we see that u^2 satisfies the problem

$$\begin{aligned} &Lu^2 = 0 \text{ in } Q, \\ &u^2(x,0) = g_s^2(x),\ u^2(x,1) = g_n^2(x) \text{ for } 0 < x < 1, \\ &u^2(0,y) = g_w^2(y),\ u^2(1,y) = g_e^2(y) \text{ for } 0 < y < 1 \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} g_s^2(x) &= -z_{01}(x,0) - z_{11}(x,0) - (1-\chi(x))W(x,0), \\ g_n^2(x) &= -z_{00}(x,1) - z_{10}(x,1) - (1-\chi(x))W(x,1), \\ g_w^2(y) &= -W(0,y) - z_{10}(0,y) - z_{11}(0,y), \\ g_e^2(y) &= -[1-\chi(1-y)]z_{00}(1,y) - [1-\chi(y)]z_{01}(1,y). \end{aligned} \tag{5.12}$$

We now consider properties of the boundary data (5.12). First, we note that using Theorem 3.14, Theorem 4.11 and Lemma 5.1, each of the functions g_s^2, g_n^2, g_w^2 and g_e^2 appearing in (5.12) is exponentially small.

Second, we assert that the data of (5.12) are compatible to arbitrary order at the 4 corners. Consider first the corner $(0,0)$. One has

$$\begin{aligned} g_s^2(x) &= -z_{01}(x,0) - z_{11}(x,0) - W(x,0) \text{ near } x = 0, \\ g_w^2(y) &= -z_{10}(0,y) - z_{11}(0,y) - W(0,y). \end{aligned}$$

Since $z_{10}(x,0) = 0$, $z_{01}(0,y) = 0$, these can be written as

$$\begin{aligned} g_s^2(x) &= -z_{01}(x,0) - z_{11}(x,0) - z_{10}(x,0) - W(x,0) \text{ near } x = 0, \\ g_w^2(y) &= -z_{10}(0,y) - z_{11}(0,y) - z_{01}(0,y) - W(0,y). \end{aligned}$$

Now each of the functions $z_{01}, z_{11}, z_{10}, W$ is smooth at $(0,0)$, and it follows that the data is compatible at $(0,0)$ to arbitrary order. A similar argument shows the compatibility at $(0,1)$. Next consider the corner $(1,0)$. One has

$$\begin{aligned} g_s^2(x) &= -z_{01}(x,0) - z_{11}(x,0) \text{ near } x = 1, \\ g_e^2(y) &= -z_{01}(1,y) \text{ near } y = 0. \end{aligned}$$

But $z_{11}(1,y) = 0$ near $y = 0$, so

$$\begin{aligned} g_s^2(x) &= -z_{01}(x,0) - z_{11}(x,0) \text{ near } x = 1, \\ g_e^2(y) &= -z_{01}(1,y) - z_{11}(1,y) \text{ near } y = 0. \end{aligned}$$

Each of the functions z_{01}, z_{11} is smooth at $(1,0)$, and it follows that the data is compatible at $(1,0)$ to arbitrary order. A similar argument shows the compatibility at $(1,1)$.

Using the fact that the data in the problem (5.11) are both exponentially small and compatible to all orders, one may show that derivatives of u^2 are bounded, uniformly in ε, in $\bar{Q}$.

From the above construction one has

$$u = U + W + z_{00} + z_{01} + z_{10} + z_{11} + u^2. \tag{5.13}$$

The derivatives of each of the terms in (5.11) have been estimated. This leads to our final theorem giving bounds for the derivatives of u.

To state the theorem, for $\lambda, \mu = 0, 1$ we set $r_{\lambda\mu} = \sqrt{(x-\lambda)^2 + (y-\mu)^2}$. Thus, $r_{\lambda\mu}$ denotes the distance from (x, y) to the vertex (λ, μ) of Q. We also let $\nu_{\lambda\mu}$ denote the compatibility of the data at the point (λ, μ). Each term $T_{\lambda\mu}$ below describes the behaviour induced in the solution by the vertex at (λ, μ); the terms $T_{0\mu}$ also include the effect of the parabolic boundary layers along $y = \mu$. The term T_E describes the effect of the exponential outflow layer at $x = 1$.

Theorem 5.2. *Let m, n be non-negative integers satisfying $2m + n \le 2\ell$ and $m + n \le 2\ell - 2$. Then for all $(x, y) \in Q$, the solution u of (1.1) satisfies:*

$$|D_x^m D_y^n u(x,y)| \le C(1 + T_{00} + T_{01} + T_{10} + T_{11} + T_E)$$

where

$$T_E = \varepsilon^{-m} e^{-p(1-x)/\varepsilon},$$

and where for $\mu = 0, 1$,

$$\begin{aligned}
T_{0\mu} &= \varepsilon^{-n/2} + \varepsilon^{\nu_{0\mu}+1-m-n} \text{ for } m+n < 2\nu_{0\mu}+2 \text{ and } r_{0\mu} < \varepsilon,\\
T_{0\mu} &= \varepsilon^{-n/2} + \varepsilon^{-\nu_{0\mu}-1}|\ln r_{0\mu}| \text{ for } m+n = 2\nu_{0\mu}+2 \text{ and } r_{0\mu} < \varepsilon,\\
T_{0\mu} &= \varepsilon^{-n/2} + \varepsilon^{-\nu_{0\mu}-1} r_{0\mu}^{2\nu_{0\mu}+2-m-n} \text{ for } m+n > 2\nu_{0\mu}+2 \text{ and } r_{0\mu} < \varepsilon,\\
T_{00} &= \varepsilon^{-n/2}\left[1 + r_{00}^{\nu_{00}+1-m-n/2}\right] e^{-cy/\sqrt{\varepsilon}} \text{ for } \varepsilon \le r_{00},\\
T_{01} &= \varepsilon^{-n/2}\left[1 + r_{01}^{\nu_{01}+1-m-n/2}\right] e^{-c(1-y)/\sqrt{\varepsilon}} \text{ for } \varepsilon \le r_{01},
\end{aligned}$$

and

$$\begin{aligned}
T_{1\mu} &= \varepsilon^{-2m-\bar{n}}\left[\varepsilon^{-n/2} + \varepsilon^{\nu_{1\mu}+1-m-n}\right] \text{ for } m+n < 2\nu_{1\mu}+2 \text{ and } r_{1\mu} < \varepsilon,\\
T_{1\mu} &= \varepsilon^{-2m-\bar{n}}\left[\varepsilon^{-n/2} + \varepsilon^{-\nu_{1\mu}-1}|\ln r_{1\mu}|\right] \text{ for } m+n = 2\nu_{1\mu}+2 \text{ and } r_{1\mu} < \varepsilon,\\
T_{1\mu} &= \varepsilon^{-2m-\bar{n}}\left[\varepsilon^{-n/2} + \varepsilon^{-\nu_{1\mu}-1} r_{1\mu}^{2\nu_{1\mu}+2-m-n}\right] \text{ for } m+n > 2\nu_{1\mu}+2 \text{ and } r_{1\mu} < \varepsilon,\\
T_{10} &= \varepsilon^{-2m-\bar{n}}\left[1 + r_{10}^{\nu_{10}+1-m-n/2}\right] e^{-p(1-x)/\varepsilon} e^{-cy/\sqrt{\varepsilon}} \text{ for } \varepsilon \le r_{10},\\
T_{11} &= \varepsilon^{-2m-\bar{n}}\left[1 + r_{11}^{\nu_{11}+1-m-n/2}\right] e^{-p(1-x)/\varepsilon} e^{-c(1-y)/\sqrt{\varepsilon}} \text{ for } \varepsilon \le r_{11}.
\end{aligned}$$

The constants C and c depend on m, n and ℓ.

Proof. We use the decomposition (5.13). Theorem 3.5 yields $\|D_x^m D_y^n U\|_{\infty,Q} \le C\|f\|_{m+n,\infty,Q} \le C\|f\|_{2\ell,\infty,Q}$. From Theorem 3.14,

$$|D_x^m D_y^n W(x,y)| \le C(\|g_e\|_{\bar{m}+n,\infty,(0,1)}) + \|U(1,\cdot)\|_{\bar{m}+n,\infty,\mathbb{R}})T_E \le C(\|g_e\|_{2\ell,\infty,(0,1)} + \|U(1,\cdot)\|_{2\ell,\infty,\mathbb{R}})T_E.$$

From Theorem 4.11, $|D_x^m D_y^n z_{00}(x,y)| \le CT_{00}$. Applying Theorem 4.11 with y replaced by $1-y$, one finds that $|D_x^m D_y^n z_{01}(x,y)| \le CT_{01}$. From Lemma 5.1, $|D_x^m D_y^n z_{10}(x,y)| \le CT_{10}$. Applying Lemma 5.1 with y

replaced by $1-y$, one finds that $|D_x^m D_y^n z_{11}(x,y)| \le CT_{11}$. To bound u^2 we note that the boundary data (5.12) of the problem (5.11) belongs to $C^{2\ell}([0,1])$. Let $G \in C^{2\ell}(\bar{Q})$ be an extension of this boundary data to all of $\bar{Q}$. Let $u^3 = u^2 - G$. Then u^3 satisfies the problem

$$Lu^3 = -LG \text{ in } Q,\ u^3 = 0 \text{ on } \Gamma. \tag{5.14}$$

The boundary data of the problem (5.14) is compatible to order $\nu = \ell$ at the 4 corners of Q, so (see the discussion in §2) $u^3 \in C^{2\ell}(\bar{Q})$. An energy argument shows that

$$\varepsilon^{1/2}\|u^3\|_{H^1(Q)} + \|u^3\|_{L^2(Q)} \le C\|LG\|_{L^2(Q)}. \tag{5.15}$$

Writing $-\Delta u^3 = \varepsilon^{-1}[LG - pu_x^3 - qu^3]$, one obtains

$$\|u^3\|_{H^2(Q)} \le C\|\varepsilon^{-1}[LG - pu_x^3 - qu^3]\|_{L^2(Q)} \le C\varepsilon^{-3/2}\|LG\|_{L^2(G)},$$

where we invoked (5.15). Continuing, one obtains

$$\|u^3\|_{H^k(Q)} \le C\varepsilon^{-(k-1/2)}\|LG\|_{L^{k-2}(G)} \text{ for } k = 0, \cdots, 2\ell.$$

Since the data (5.12) are exponentially small, we obtain $\|u^3\|_{H^{2\ell}(Q)} \le C$, and from Sobolev's inequality, $\|u^3\|_{2\ell-2,\infty,Q} \le C$. ■

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