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Abstract

We survey results concerning the maximum size of a family $\mathcal{F}$ of subsets of an n-element set such that a certain configuration is avoided. When $\mathcal{F}$ avoids a chain of size two, this is just Sperner's Theorem. Here we give bounds on how large $\mathcal{F}$ can be such that no four distinct sets $A, B, C, D \in \mathcal{F}$ satisfy $A \subset B,\ C \subset B,\ C \subset D$. In this case, the maximum size satisfies $\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) \leq |\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right)$, which is very similar to the best-known bounds for the more restrictive problem of $\mathcal{F}$ avoiding three sets B, C, D such that $C \subset B,\ C \subset D$.

1 Introduction

Let $[n] = \{1, 2, \dots, n\}$ be a finite set, $\mathcal{F} \subset 2^{[n]}$ a family of its subsets. In the present paper $\max |\mathcal{F}|$ will be investigated under certain conditions on the family $\mathcal{F}$. The well-known Sperner's Theorem ([7]) was the first such discovery.

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Theorem 1.1 *If $\mathcal{F}$ is a family of subsets of $[n]$ without inclusion ($F, G \in \mathcal{F}$ implies $F \not\subset G$) then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

holds, and this estimate is sharp as the family of all $\lfloor \frac{n}{2} \rfloor$-element subsets shows.

There is a very large number of generalizations and analogues of this theorem. Here we will mention only some results when the condition on $\mathcal{F}$ excludes certain configurations that can be expressed by inclusion only. That is, no intersections, unions, etc. are involved. The first such generalization was obtained by Erdős [3]. The family of k distinct sets with mutual inclusions, $F_1 \subset F_2 \subset \ldots F_k$ is called a *chain of length k*, which we denote simply by P_k. For any family of sets F, with specified inclusions between pairs of sets, let $\mathrm{La}(n, F)$ denote the size of the largest family $\mathcal{F}$ of subsets of $[n]$ without any F. Erdős extended Sperner's Theorem as follows:

Theorem 1.2 [3] $\mathrm{La}(n, P_{k+1})$ *is equal to the sum of the k largest binomial coefficients of order n.*

Now consider families other than chains. Let V_r denote the *r-fork*, which is a family of $r+1$ distinct sets: $F \subset G_1, F \subset G_2, \ldots, F \subset G_r$. The quantity $\mathrm{La}(n, V_r)$ was first (asymptotically) determined for $r = 2$.

Theorem 1.3 [5]

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) \leq \mathrm{La}(n, V_2) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{2}{n}\right).$$

This was recently generalized:

Theorem 1.4 [1], cf. [8]

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{r}{n} + O\left(\frac{1}{n^2}\right)\right) \leq \mathrm{La}(n, V_{r+1}) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right).$$

Four distinct subsets satisfying $A \subset C, A \subset D, B \subset C, B \subset D$ are called a *butterfly* and are denoted by B.

Theorem 1.5 [2] *Let* $n \geq 3$. *Then* $\mathrm{La}(n, B) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$.

Let us compare Theorems 1.3 and 1.5. When V_2 is excluded, then the largest family has size equal to that of the largest level plus between $\frac{1}{n}$ and $\frac{2}{n}$ times the next level. On the other hand, if the butterfly is excluded then the size of two levels can be achieved. It is a natural question, what happens if a configuration between these two is excluded. Namely, let the configuration of four distinct subsets satisfying $A \subset C, A \subset D, B \subset C$ be called and denoted by N. It is somewhat surprising that the result is basically the same (at least in the first two terms) as in the case of V_2. The total "jump" is between N and B. The goal of the present paper is to prove the following theorem.

Theorem 1.6

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) \leq La(n, N) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right)$$

holds.

2 Notation and definitions

A *partially ordered set* (or *poset*) P is a pair $P = (X, \leq)$ where X is a set (in our case always finite) and $\leq$ is a relation on X which is *reflexive* ($x \leq x$ holds for every $x \in X$), *antisymmetric* (if both $x \leq y$ and $x \geq y$ hold for $x, y \in X$ then $x = y$) and *transitive* ($x \leq y$ and $y \leq z$ always implies $x \leq z$). It is easy to see that if $X = 2^{[n]}$ and the $\leq$ is defined as $\subseteq$, then these conditions are satisfied, that is, the family of all subsets of an n-element set ordered by inclusion form a poset. We will call this poset the *Boolean lattice* and denote it by B_n.

The definition of a *subposet* is obvious: $R = (Y, \leq_2)$ is a subposet of $P = (X, \leq_1)$ if and only if there is an injection α of Y into X in such a way that $y_1, y_2 \in Y, y_1 \leq_2 y_2$ implies $\alpha(y_1) \leq_1 \alpha(y_2)$. On the other hand, R is an *induced subposet* of P when $\alpha(y_1) \leq_1 \alpha(y_2)$ holds if and only if $y_1 \leq_2 y_2$. If $P = (X, \leq)$ is a poset and $Y \subset X$ then the poset *spanned by* Y *in* P is defined as $(Y, \leq^*)$ where $\leq^*$ agreers with $\leq$, for all the pairs taken from Y. Given a "small" poset R, let $\mathrm{La}(n, R)$ denote the maximum size $|Y|$ for $Y \subset 2^{[n]}$ (that is, the maximum number of subsets of $[n]$) such that R is not a subposet of the poset spanned by Y in B_n.

Redefine our “small” configurations in terms of posets. The chain P_k contains k elements: $a_1, \ldots, a_k$ where $a_1 < \ldots < a_k$. The *r-fork* contains $r+1$ elements: $a, b_1, \ldots, b_r$ where $a < b_1, \ldots a < b_r$. The butterfly B contains 4 elements: a, b, c, d with $a < c, a < d, b < c, b < d$. Finally, we have the following relations in N: $a < c, a < d, b < c$. It is easy to see that the definitions of $\mathrm{La}(n, P_k)$, $\mathrm{La}(n, V_r)$, $\mathrm{La}(n, B)$ and $\mathrm{La}(n, N)$ in Sections 1 and 2 agree. In the rest of the paper we will use both sets of terminology.

A poset is *connected* if for any pair (z_0, z_k) of its elements there is a sequence $z_1, \ldots, z_{k-1}$ such that either $z_i < z_{i+1}$ or $z_i > z_{i+1}$ holds for $0 \leq i < k$. If the poset is not connected, maximal connected subposets are called its *connected components*. Given a family $\mathcal{F}$ of subsets of $[n]$, it spans a poset in B_n. We will consider its connected components in two different ways. First as posets themselves, secondly as they are represented in B_n. In the latter case the sizes of the sets are also indicated. A *full chain* in B_n is a family of sets $A_0 \subset A_1 \subset \ldots \subset A_n$ where $|A_i| = i$. Let us mention that the number of full chains in B_n is $n!$. We say that a (full) chain *goes through* a family (subposet) $\mathcal{F}$ if they intersect, that is, if the chain “goes through” at least one member of the family.

3 Proof of Theorem 1.6

The lower estimate is obtained from Theorem 1.3, since $\mathrm{La}(n, V_2) \leq \mathrm{La}(n, N)$.

The upper estimate uses an idea generalizing the proof of Sperner’s Theorem given by Lubell [6], which is based on counting the number of full chains passing through a family. Let $\mathcal{F}$ be a family of subsets of $[n]$ containing no four distinct members forming an N. Consider the poset $P(\mathcal{F})$ spanned by $\mathcal{F}$ in B_n. Its connected components are denoted by $\mathcal{F}_1, \ldots, \mathcal{F}_K$. Let $c(\mathcal{F}_i)$ denote the number of full chains going through $\mathcal{F}_i$. Observe that a full chain cannot go through two distinct components. Otherwise, they would be the same component. Therefore, the following inequality holds:

$$\sum_{i=1}^{K} c(\mathcal{F}_i) \leq n!. \tag{1}$$

What can these components be? A component might be a P_3, but one can check by cases that no component can contain a P_3 as a proper subposet, since adding one more element to P_3 creates an N, no matter which element of P_3 is related to the new element. Hence, if $a < b$ are two elements of a component of size at least four, then a and b cannot be both comparable with the same other element of the component (which would create P_3), though one of a, b can be comparable with many other elements in the component. Therefore, the only possible components are these:

$$a < b < c, \tag{2}$$

$$a < b_i (1 \leq i \leq r) \text{ where } r \geq 0, \tag{3}$$

$$a > b_i (1 \leq i \leq r) \text{ where } r \geq 2. \tag{4}$$

These are denoted by $P(3), V(r), \Lambda(r)$ in this order. (The elements b_i are unrelated here.) Notice that the number of full chains going through a poset of these types depends only on the sizes of the elements of the poset, that is, the sizes of the members of the family $\mathcal{F}_i$. To indicate this size information, we introduce the notation $P(3; u, v, w)$ for posets of type $P(3)$ where $|a| = u < |b| = v < |c| = w$. The analogous notations for the other poset types above are are $V(r; u, u_1, \ldots, u_r)$ $(u < u_1, \ldots, u_r)$ and $\Lambda(r; v, v_1, \ldots, v_r)$ $(v > v_1, \ldots, v_r)$. All components $\mathcal{F}_i$ in (1) are of this form.

Plan of the proof. A good upper bound is sought for

$$|\mathcal{F}| = \sum_{i=1}^{K} |\mathcal{F}_i| \tag{5}$$

where $|P(3)| = 3, |V(r)| = |\Lambda(r)| = r + 1$ are obvious. This upper bound will be determined entirely on the basis of (1). Denote the numbers of F_is of types $P(3), V(r), \Lambda(r)$ by $\phi, \nu(r), \lambda(r)$ respectively. Then (5) can be written in the form

$$3\phi + \sum_{r=0}^{\infty} (r+1)\nu(r) + \sum_{r=2}^{\infty} (r+1)\lambda(r). \tag{6}$$

If the minima (or good lower bounds)

$$\min_{u,v,w} c(P(3; u, v, w)), \min_{u,u_1,\ldots,u_r} c(V(r; u, u_1, \ldots, u_r)), \min_{v,v_1,\ldots,v_r} c(\Lambda(r; v, v_1, \ldots, v_r))$$

are determined then (1) leads to a linear combination of $\phi, \nu(r)$, and $\lambda(r)$. That is, one linear combination, namely (5), has to be maximized under the condition that another combination is bounded from above. This will be an easy task. Therefore, our main problem now is to determine the minima in the display above. This will be done in the following two lemmas.

Lemma 3.1 *$c(P(3;u,v,w))$ $(u<v<w)$ takes its minimum for the values $u=\lfloor\frac{n}{2}\rfloor-1, v=\lfloor\frac{n}{2}\rfloor, w=\lfloor\frac{n}{2}\rfloor+1$, that is,*

$$\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)!\left(\left\lceil\frac{n}{2}\right\rceil-1\right)!\left(\left\lfloor\frac{n}{2}\right\rfloor^2-n\left\lfloor\frac{n}{2}\right\rfloor+n^2-1\right)\le c(P(3;u,v,w)).$$

Proof. The number of full chains going through $P(3;u,v,w)$ is

$$c(P(3;u,v,w)) = u!(n-u)! + v!(n-v)! + w!(n-w)!$$
$$-u!(v-u)!(n-v)! - v!(w-v)!(n-w)! - u!(w-u)!(n-w)! \qquad (7)$$
$$+u!(v-u)!(w-v)!(n-w)!$$

by the ordinary sieve. Since this expression has the same value when we replace u, v, w by $n-u, n-v, n-w$, respectively, $\frac{n}{2} \le v$ can be supposed. Another useful form of (7) is

$$\frac{c(P(3;u,v,w))}{n!} = \frac{1}{\binom{n}{u}} + \frac{1}{\binom{n}{v}} + \frac{1}{\binom{n}{w}} - \frac{1}{\binom{n}{v}\binom{v}{u}} - \frac{1}{\binom{n}{w}\binom{w}{v}} - \frac{1}{\binom{n}{w}\binom{w}{u}} + \frac{1}{\binom{n}{w}\binom{w}{v}\binom{v}{u}}$$

$$= \frac{1}{\binom{n}{u}} + \frac{1}{\binom{n}{v}} - \frac{1}{\binom{n}{v}\binom{v}{u}} + \frac{1}{\binom{n}{w}}\left(1 - \frac{1}{\binom{w}{u}} - \frac{1}{\binom{w}{v}}\left(1 - \frac{1}{\binom{v}{u}}\right)\right). \qquad (8)$$

Here $\binom{n}{w}$ is a decreasing function of w in the interval $[v, n]$ (with fixed u, v). On the other hand, $\binom{w}{u}$ and $\binom{w}{v}$ are increasing functions. Therefore, (8) is smallest for $w = v+1$: One has to see that the coefficients of $1/\binom{n}{w}$ and $1/\binom{w}{v}$ are nonnegative, but this is an easy task if $0 < u$, and, otherwise, (8) is equal to 1.

w can be replaced by $v+1$ in (7):

$$c(P(3;u,v,v+1)) = u!(n-u)! + nv!(n-v-1)! - u!(v-u)!(n-v-1)!(n-u). \qquad (9)$$

Another useful form of (9) is

$$\frac{c(P(3;u,v,v+1))}{n!} = \frac{1}{\binom{n}{u}} + \frac{1}{\binom{n-1}{v}}\left(1 - \frac{n-u}{n} \cdot \frac{1}{\binom{v}{u}}\right). \qquad (10)$$

Here $\binom{n-1}{v}$ is a decreasing function of v in the interval $[\lfloor \frac{n-1}{2} \rfloor, n-1]$, while $\binom{v}{u}$ is increasing. Therefore, if we want to minimize (10), v can be chosen as the smallest integer $\ge \lfloor \frac{n}{2} \rfloor$ with $v > u$. Two cases will be distinguished: $v = u+1$ and $v = \lfloor \frac{n}{2} \rfloor$.

Case 1. $v = u+1$. In this case (9) becomes

$$c(P(3;v-1,v,v+1))$$

$$= (v-1)!(n-v+1)! + nv!(n-v-1)! - (v-1)!(n-v-1)!(n-v+1)$$

$$= (v-1)!(n-v-1)!(v^2-nv+n^2-1) = \frac{(n-2)!}{\binom{n-2}{v-1}}(v^2-nv+n^2-1).$$

It is easy to see that both factors attain their respective minima at $v = \lfloor \frac{n}{2} \rfloor$. The lemma is proved for this case.

Case 2. $v = \lfloor \frac{n}{2} \rfloor$. Rewrite (10):

$$\frac{c(P(3; u, v, v+1))}{n!} = \frac{1}{\binom{n-1}{v}} + \frac{1}{\binom{n}{u}} \left(1 - \frac{1}{\binom{n-u-1}{n-v-1}} \right).$$

$\binom{n}{u}$ is an increasing function of u in the interval $[0, \lfloor \frac{n}{2} \rfloor]$ while $\binom{n-u-1}{n-v-1}$ is decreasing. The best choice for u is $\lfloor \frac{n}{2} \rfloor - 1$. $\square_{\mathrm{L}}$

Lemma 3.2 *Suppose* $n \geq 6, r \geq 1$. *Then*

$$u^*!(n-u^*)! + ru^*!u^*(n-u^*-1)! \leq c(V(r; u, u_1, \ldots, u_r)) \qquad (u < u_1, \ldots, u_r)$$

holds where $u^* = u^*(n) = \frac{n}{2} - 1$ *if* n *is even,* $u^* = \frac{n-1}{2}$ *if* n *is odd and* $r - 1 \leq n$, *while* $u^* = \frac{n-3}{2}$ *if* n *is odd and* $n < r - 1$.

Proof.

1. One can easily show by using the sieve that

$$c(V(r; u, u_1, \ldots, u_r)) = u!(n-u)! + \sum_{i=1}^{r} u_i!(n-u_i)! - \sum_{i=1}^{r} u!(u_i-u)!(n-u_i)!.$$

This will actually be used in the form

$$c(V(r; u, u_1, \ldots, u_r)) = \sum_{i=1}^{r} \left(\frac{1}{r} u!(n-u)! + u_i!(n-u_i)! - u!(u_i-u)!(n-u_i)! \right). \tag{11}$$

Dividing one term by $n!$, two useful forms are obtained for the summand in (11):

$$\frac{1}{r\binom{n}{u}} + \frac{1}{\binom{n}{u_i}} - \frac{1}{\binom{n}{u_i}\binom{u_i}{u}} = \frac{1}{r\binom{n}{u}} + \frac{1}{\binom{n}{u_i}} \left(1 - \frac{1}{\binom{u_i}{u}} \right) \tag{12}$$

and

$$\frac{1}{r\binom{n}{u}} + \frac{1}{\binom{n}{u_i}} - \frac{1}{\binom{n}{u}\binom{n-u}{n-u_i}} = \frac{1}{\binom{n}{u_i}} + \frac{1}{\binom{n}{u}}\left(\frac{1}{r} - \frac{1}{\binom{n-u}{n-u_i}}\right). \qquad (13)$$

2. First we will show that (12)-(13) attains its minimum for some pair $u, u_i = u+1$.

If $\frac{n}{2} - 1 \leq u$, fix u and consider changing u_i in (12). Here, $\binom{n}{u_i}$ is a decreasing function of u_i in the interval $[\lfloor \frac{n}{2} \rfloor, n]$, while $\binom{u_i}{u}$ is increasing. Therefore, one can suppose that $u_i = u+1$, and we are done.

Else, $\frac{n}{2} - 1 > u$ and the method in (12) above leads to $u_i \leq \lfloor \frac{n}{2} \rfloor$. Fix this value and increase u using (13). It will not increase by moving u to $u = \lfloor \frac{n}{2} \rfloor - 1$.

Hence, we obtained the lower estimate

$$\min_u \left(\frac{1}{r} u!(n-u)! + (u+1)!(n-u-1)! - u!1!(n-u-1)!\right) = \min_u \left(\frac{1}{r} u!(n-u)! + u!u(n-u-1)!\right)$$

for (12)-(13) and therefore we have

$$\min_u \left(u!(n-u)! + ru!u(n-u-1)!\right) \leq c(V(r; u, u_1, \ldots, u_r)) \qquad (14)$$

This minimum will be determined in the rest of the proof.

3. Suppose now that $2 \leq r$. Take the "derivative" of $f_r(u) = u!(n-u)! + ru!u(n-u-1)!$, that is, compare $f_r(u)$ at two consecutive values of u. When does the inequality

$$\begin{aligned} f_r(u-1) = (u-1)!(n-u+1)! + r(u-1)!(u-1)(n-u)! < \\ f_r(u) = u!(n-u)! + ru!u(n-u-1)! \end{aligned} \qquad (15)$$

hold? It is equivalent to

$$0 < 2(r-1)u^2 - \big(n(r-3) + r - 1\big)u - n^2 + (r-1)n.$$

The discriminant of the corresponding quadratic equation in u is

$$\big(n(r-3)+r-1\big)^2 + 8(r-1)\big(n^2-(r-1)n\big) = (r+1)^2n^2 - 2(r-1)(3r-1)n + (r-1)^2.$$

The latter expression can be strictly upperestimated by

$$((r+1)n-(r-1))^2\,,$$

if $r+1<3r-1$ holds, that is, if $r>1$. Hence, the larger root α_2 of the quadratic equation is less than

$$\frac{n(r-3)+r-1+(r+1)n-(r-1)}{4(r-1)}=\frac{n}{2}.$$

On the other hand, as it is easy to see, $(n(r+1)-3(r-1))^2$ is a lower bound for the discriminant if $r-1\le n$ holds. Using this estimate, we obtain that $\frac{n-1}{2}\le\alpha_2$ in this case. Substituting this lower estimate into the formula for the smaller root α_1, we obtain $\alpha_1\le 0$ when $n\ge r-1$. Since (15) holds exactly below α_1 and above α_2, we can state that $f_r(u)$ attains its minimum at $u=\lfloor\alpha_2\rfloor$. By the inequalities above we can conclude that this is at $\frac{n}{2}-1$ if n is even and $\frac{n-1}{2}$ if n is odd. The statement of the lemma is proved in the case of $n\ge r-1$.

Else suppose $n<r-1$. The inequality $\alpha_2<\frac{n-1}{2}$ can be proved in the same way as in the previous case. On the other hand, $6\le n$ implies that $(n(r+1)-5(r-1))^2$ is a lower estimate on the discriminant, hence we have $\frac{n}{2}-1<\alpha_2$. This gives that $\alpha_1<\frac{3}{2}$. The if n is even, $\lfloor\alpha_2\rfloor$ is again $\frac{n}{2}-1$, while $\lfloor\alpha_2\rfloor=\frac{n-3}{2}$ when n is odd. Although $f_r(0)<f_r(1)$ is allowed by this estimate, it is easy to check that $f_r(0)>f_r(1)$ holds in reality. By (14) the proof is finished for $r\ge 2$.

The case $r=1$ is much easier. The comparison (15) leads to a linear inequality which is an equality for $u=\frac{n}{2}$. The formula $f_1(u)$ also has its minimum at $\left\lfloor\frac{n-1}{2}\right\rfloor$. (But it has the same value at $\frac{n}{2}-1$ and $\frac{n}{2}$.) $\square_{\mathrm{L}}$

The case $r=0$ has to be treated differently. Then the number of full chains going through the only (u-element) set is $u!(n-u)!$. Its minimum is $\lfloor\frac{n}{2}\rfloor!\lceil\frac{n}{2}\rceil!$.

Introduce the notation $m(n)=u^*!(n-u^*)!+ru^*!u^*(n-u^*-1)!$ where $u^*=u^*(n)$ is determined in Lemma 3.2.

Proof of the Theorem. We will need the following inequalities in the proof:

$$u^*!u^*(n-u^*-1)!<\frac{u^*!(n-u^*)!+ru^*!u^*(n-u^*-1)!}{r+1} \tag{16}$$

holds for each of the 3 possible values of u^*. They easily reduce to $\frac{n}{2}-1<\frac{n}{2}+1$, $\frac{n-1}{2}<\frac{n+1}{2}$ and $\frac{n-3}{2}<\frac{n+3}{2}$ when $u^*=\frac{n}{2}-1$, $\frac{n-1}{2}$ and $\frac{n-3}{2}$, respectively.

A different inequality is needed in the case $r = 0$:

$$u^*!u^*(n-u^*-1)! < \left\lfloor \frac{n}{2} \right\rfloor ! \left\lceil \frac{n}{2} \right\rceil !. \tag{17}$$

One can verify it separating the 3 cases.

Similarly

$$\begin{aligned} u^*!u^*(n-u^*-1)! \leq \\ \frac{1}{3}\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)! \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)! \left(\left\lfloor \frac{n}{2} \right\rfloor^2 - n\left\lfloor \frac{n}{2} \right\rfloor + n^2 - 1\right) \end{aligned} \tag{18}$$

can be obtained for each case.

Start the proof by (1):

$$n! \geq \sum_{i=1}^{K} c(\mathcal{F}_i) = \sum_{i=1}^{K} \frac{c(\mathcal{F}_i)}{|\mathcal{F}_i|}|\mathcal{F}_i| \tag{19}$$

If $\mathcal{F}_i$ is a $P(3)$, then Lemma 3.1 and (18) give

$$\frac{c(\mathcal{F}_i)}{|\mathcal{F}_i|} = \frac{c(\mathcal{F}_i)}{3} \geq u^*!u^*(n-u^*-1)!.$$

If $\mathcal{F}_i$ is an $V(r)$ then Lemma 3.2 and (16) prove

$$\frac{c(\mathcal{F}_i)}{|\mathcal{F}_i|} = \frac{c(\mathcal{F}_i)}{r+1} \geq u^*!u^*(n-u^*-1)!$$

for $1 \leq r$ and (17) shows its validity when $r = 0$. By symmetry one can say the same about $\Lambda(r)$. Display (19) results in

$$\begin{aligned} n! \geq \sum_{i=1}^{K} c(\mathcal{F}_i) = \sum_{i=1}^{K} \frac{c(\mathcal{F}_i)}{|\mathcal{F}_i|}|\mathcal{F}_i| \geq \\ u^*!u^*(n-u^*-1)! \sum_{i=1}^{K} |\mathcal{F}_i| = u^*!u^*(n-u^*-1)!|\mathcal{F}|. \end{aligned}$$

Hence we obtained

$$|\mathcal{F}| \leq \frac{n!}{u^*!u^*(n-u^*-1)!},$$

and this is equal to

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{\frac{n}{2}}{\frac{n}{2} - 1}, \qquad \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{\frac{n+1}{2}}{\frac{n-1}{2}}, \qquad \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{\frac{n-1}{2}}{\frac{n-3}{2}}$$

in the cases $u^* = \frac{n}{2} - 1, \frac{n-1}{2}$ and $\frac{n-3}{2}$, respectively. These are all equal to

$$= \binom{n}{\lfloor \frac{n}{2} \rfloor} \Big(1 + \frac{2}{n} + O\Big(\frac{1}{n^2}\Big)\Big).$$

$\square_{\mathrm{T}}$

4 Remark

It is somewhat disturbing that the constant in the second term in Theorem 1.6 (and (1.3)) is not fully determined. Is it 1 or 2? Let us make the situation clear.

The construction of a family avoiding V_2 is the following: Take all the sets of size $\lfloor \frac{n}{2} \rfloor$ and a family $A_1, \dots A_m$ of $\lfloor \frac{n}{2} \rfloor + 1$-element sets satisfying the condition $|A_i \cap A_j| < \lfloor \frac{n}{2} \rfloor$ for every pair $i < j$. It is easy to see that this family contains no V_2. We only have to maximize m. Denote this maximum by $m(n)$. Since the $\lfloor \frac{n}{2} \rfloor$-element subsets of the A_is are all distinct, we have

$$m \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

This gives the upper estimate

$$m(n) \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{2}{n}.$$

There is a very nice construction (see [4]) of such sets A_i with

$$m = \binom{n}{\lfloor \frac{n}{2} \rfloor + 1} \frac{1}{n}.$$

We obtained

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \left(\frac{1}{n} + O\left(\frac{1}{n^2} \right) \right) \le m(n) \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(\frac{2}{n} \right). \tag{20}$$

It is a longstanding conjecture of coding theory what the right constant is here, 1 or 2. Does this limit exist at all?

It is easy to see that

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} + m(n) \leq \mathrm{La}(n, N).$$

The upper estimate of our Theorem 1.6 is a (far-reaching) generalization (up to the second term) of the upper estimate in (20). Replacing the constant 2 in the upper estimate in Theorem 1.6 by 1 would improve the upper estimate in (20), too, solving the old coding problem. On the other hand, if one could construct a family with a constant 2 rather than 1, this would improve the lower estimate in Theorem 1.6. Summarizing, there is a strong connection between the two problems. Of course, it is possible that the proper constant in $m(n)$ is 1, while the one in $\mathrm{La}(n, N)$ is 2.

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