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# DECOMPOSITION OF TRIEBEL-LIZORKIN AND BESOV SPACES IN THE CONTEXT OF LAGUERRE EXPANSIONS

G. KERKYACHARIAN, P. PETRUSHEV, D. PICARD, AND YUAN XU

ABSTRACT. A pair of dual frames with almost exponentially localized elements (needlets) are constructed on  $\mathbb{R}_+^d$  based on Laguerre functions. It is shown that the Triebel-Lizorkin and Besov spaces induced by Laguerre expansions can be characterized in terms of respective sequence spaces that involve the needlet coefficients.

## 1. INTRODUCTION

The primary goal of this paper is to construct frames on  $\mathbb{R}_+^d := (0, \infty)^d$  with nearly exponentially localized elements, based on Laguerre functions and utilize them to the characterization of spaces of distribution on  $\mathbb{R}_+^d$ . We are interested in extending the fundamental results of Frazier and Jawerth [5, 6, 7] on the  $\varphi$ -transform on  $\mathbb{R}^d$  in the context of Laguerre expansions.

From the three types of Laguerre functions available in the literature we focus our attention on the Laguerre functions  $\{\mathcal{F}_\nu^\alpha\}$  (see (3.1)) which form an orthonormal basis for the space  $L^2(\mathbb{R}_+^d, w_\alpha)$  with weight

$$(1.1) \quad w_\alpha(x) := \prod_{j=1}^d x_j^{2\alpha_j+1}.$$

For various technical reasons we will assume that  $\alpha_j \geq 0$ , while in general  $\alpha_j > -1$ . The other two classes of Laguerre functions  $\{\mathcal{L}_\nu^\alpha\}$  and  $\{\mathcal{M}_\nu^\alpha\}$  (see (3.4)-(3.5)) form orthogonal bases for  $L^2(\mathbb{R}_+^d)$  (weight 1). The  $d$ -dimensional Laguerre functions  $\mathcal{F}_\nu^\alpha$  are products of univariate Laguerre functions, namely,  $\mathcal{F}_\nu^\alpha(x) := \mathcal{F}_{\nu_1}^\alpha(x_1) \cdots \mathcal{F}_{\nu_d}^\alpha(x_d)$  (see (3.1), (3.3)). Hence the kernel of the orthogonal projector onto

$$(1.2) \quad W_n := \text{span}\{\mathcal{F}_\nu^\alpha : |\nu| = n\} \quad \text{is given by} \quad \mathcal{F}_n^\alpha(x, y) := \sum_{|\nu|=n} \mathcal{F}_\nu^\alpha(x) \mathcal{F}_\nu^\alpha(y).$$

Denote  $V_n := \bigoplus_{m=0}^n W_m$ . Evidently,  $K_n(x, y) := \sum_{m=0}^n \mathcal{F}_m^\alpha(x, y)$  is the kernel of the orthogonal projector onto  $V_n$ . A main point in the present paper is that for compactly supported  $C^\infty$  cut-off functions  $\hat{a}$  which are constant around zero the kernels

$$(1.3) \quad \Lambda_n(x, y) := \sum_{j=0}^{\infty} \hat{a}\left(\frac{j}{n}\right) \mathcal{F}_j^\alpha(x, y)$$

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decay rapidly (almost exponentially) away from the main diagonal in  $\mathbb{R}_+^d$  (Theorem 3.2). For the same kind of kernels associated with the Laguerre functions  $\{\mathcal{M}_\nu^\alpha\}$  in dimension  $d = 1$  this fact is established in [4]. We show that similar results are valid for  $\{\mathcal{M}_\nu^\alpha\}$  and  $\{\mathcal{L}_\nu^\alpha\}$  in dimension  $d > 1$  as well.

We utilize the kernels from (1.3) to the construction of a pair of dual frames  $\{\varphi_\xi\}_{\xi \in \mathcal{X}}$  and  $\{\psi_\xi\}_{\xi \in \mathcal{X}}$  with  $\mathcal{X}$  a multilevel index set. As in other similar settings, the almost exponential localization of  $\varphi_\xi$  and  $\psi_\xi$  prompts us to call them “needlets”. The needlet systems from this paper can be regarded as analogues of the  $\varphi$ -transform of Frazier and Jawerth [5, 6]. They are particularly well suited for characterization of the Triebel-Lizorkin and Besov spaces associated with Laguerre expansions. To be more precise, let  $\widehat{a} \in C^\infty$ ,  $\text{supp } \widehat{a} \subset [1/4, 4]$ , and  $|\widehat{a}| > c$  on  $[1/3, 3]$  and define

$$\Phi_0(x, y) := \mathcal{F}_0^\alpha(x, y) \quad \text{and} \quad \Phi_j(x, y) := \sum_{m=0}^{\infty} \widehat{a}\left(\frac{m}{4^{j-1}}\right) \mathcal{F}_m^\alpha(x, y), \quad j \geq 1,$$

Then for all appropriate indices (see Definition 6.1) the Laguerre-Triebel-Lizorkin space  $F_{pq}^{s\rho}$  is defined as the set of all tempered distributions  $f$  on  $\mathbb{R}_+^d$  such that

$$\|f\|_{F_{pq}^{s\rho}} := \left\| \left( \sum_{j=0}^{\infty} \left[ 2^{sj} W_\alpha(4^j; \cdot)^{-\rho/d} |\Phi_j * f(\cdot)| \right]^q \right)^{1/q} \right\|_p < \infty.$$

Here  $\Phi_j * f(x) := \langle f, \overline{\Phi_j(x, \cdot)} \rangle$  (Definition 4.2) and the weight  $W_\alpha(n; x)$  is define by

$$(1.4) \quad W_\alpha(n; x) := \prod_{j=1}^d (x_j + n^{-1/2})^{2\alpha_j + 1}.$$

Just for convenience we use dilations by factors of  $4^j$  on the frequency side as opposed to the traditional binary dilation. The Laguerre-Besov spaces are defined by the (quasi-)norm

$$\|f\|_{B_{pq}^{s\rho}} := \left( \sum_{j=0}^{\infty} \left( 2^{sj} \|W_\alpha(4^j; \cdot)^{-\rho/d} \Phi_j * f(\cdot)\|_p \right)^q \right)^{1/q}.$$

Unlike in the classical case on  $\mathbb{R}^d$  the weight  $w_\alpha$  creates some inhomogeneity which compels us to introduce the additional term  $W_\alpha(4^j; \cdot)^{-\rho/d}$  with parameter  $\rho \in \mathbb{R}$ . This allows to consider different scales of Triebel-Lizorkin and Besov spaces. For instance, a “classical” choice would be  $\rho = 0$ . However, more natural to us are the spaces  $F_{pq}^{ss}$  and  $B_{pq}^{ss}$  which embed “correctly” with respect to the smoothness parameter  $s$ .

The main results in this article assert that the Laguerre Triebel-Lizorkin and Besov spaces can be characterized in terms of respective sequence spaces involving the needlet coefficients of the distributions (Theorems 6.7, 7.4).

Along the same lines one can develop a similar theory on  $\mathbb{R}_+^d$  with weight 1 using the Laguerre functions  $\{\mathcal{L}_\nu^\alpha\}$  or  $\{\mathcal{M}_\nu^\alpha\}$ . For such spaces induced by  $\{\mathcal{L}_\nu^\alpha\}$ , see [2].

This paper is an integral part of a broader undertaking for needlet characterization of Triebel-Lizorkin and Besov spaces on nonstandard domains (and with weights) such as the sphere [12], interval [9], ball [10], and in the setting of Hermite expansions [14].

The outline of the paper is as follows. All the information we need about Laguerre polynomials and functions is given in §2. The localized kernels induced by Laguerre

functions are given in §3. Some additional background material is collected in §4. The construction of needlets is given in §5. In §6 the Laguerre-Triebel-Lizorkin spaces are introduced and characterized in terms of needlet coefficients, while the characterization of the Laguerre-Besov spaces is given in §7. Some proofs for Sections 3 - 4 are given in §8 and for Sections 5 - 6 in §9.

The following notation will be used throughout:  $\|x\| := \max_i |x_i|$ ,  $|x| := \sum_{i=1}^d |x_i|$ ,  $\|x\|_2 := \left( \sum_{i=1}^d |x_i|^2 \right)^{1/2}$ ,  $\|f\|_p := \left( \int_{\mathbb{R}_+^d} |f(x)|^p w_\alpha(x) dx \right)^{1/p}$ ;  $|E|$  stands for the Lebesgue measure of  $E \subset \mathbb{R}_+^d$ ,  $\mu(E) := \int_E w_\alpha(x) dx$ ,  $\mathbb{1}_E$  is the characteristic function of  $E$ , and  $\tilde{\mathbb{1}}_E := \mu(E)^{-1/2} \mathbb{1}_E$ . Positive constants are denoted by  $c, c_1, c_*, \dots$  and they may vary at every occurrence;  $A \sim B$  means  $c_1 A \leq B \leq c_2 A$ .

## 2. BACKGROUND: LAGUERRE POLYNOMIALS AND FUNCTIONS

In this section we collect the information on Laguerre polynomials and functions that will be needed in this paper. The Laguerre polynomials  $L_n^\alpha$  ( $\alpha > -1$ ) can be defined by their generating function

$$\sum_{n=0}^{\infty} L_n^\alpha(x) r^n = (1-r)^{-\alpha-1} e^{-xr/(1-r)}, \quad |r| < 1.$$

They are orthogonal on  $\mathbb{R}_+ = (0, \infty)$  with weight  $x^\alpha e^{-x}$ , more precisely,

$$\int_0^\infty L_n^\alpha(x) L_m^\alpha(x) e^{-x} x^\alpha dx = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \delta_{n,m} = \Gamma(\alpha+1) L_n^\alpha(0) \delta_{n,m},$$

where we used that  $L_n^\alpha(0) = \binom{n+\alpha}{n}$  [16, (5.1.1)].

Let  $L_\nu^\alpha(x) := L_{\nu_1}^{\alpha_1}(x_1) \cdots L_{\nu_d}^{\alpha_d}(x_d)$  be the product Laguerre polynomials on  $\mathbb{R}_+^d$ , where  $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}_0^d$  and  $\alpha = (\alpha_1, \dots, \alpha_d)$ . For  $\delta > -1$ , define

$$(2.1) \quad P_n^{\alpha, \delta}(x; y) := \sum_{k=0}^n A_{n-k}^\delta \sum_{|\nu|=k} \frac{L_\nu^\alpha(x) L_\nu^\alpha(y)}{L_\nu^\alpha(0)}, \quad A_m^\delta := \binom{m+\delta}{m}.$$

This is a constant multiple of the  $n$ th Cesàro sum of the reproducing kernels for Laguerre polynomials in dimension  $d$ . Using the generating function of the Laguerre polynomials, it is shown in [19] that

$$(2.2) \quad P_n^{\alpha, \delta}(x, 0) = L_n^{|\alpha|+\delta+d}(|x|).$$

The product formula for Laguerre polynomials (Hardy-Watson) [17, Proposition 6.1.1] asserts that: For  $\alpha > -\frac{1}{2}$  and  $x, y \in \mathbb{R}_+$ ,

$$(2.3) \quad \begin{aligned} & \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} L_n^\alpha(x^2) L_n^\alpha(y^2) \\ &= \frac{2^\alpha}{\sqrt{2\pi}} \int_0^\pi L_n^\alpha(x^2 + y^2 + 2xy \cos \theta) e^{-xy \cos \theta} j_{\alpha-1/2}(xy \sin \theta) \sin^{2\alpha} \theta d\theta, \end{aligned}$$

where  $j_\alpha(x) := x^{-\alpha} J_\alpha(x)$  with  $J_\alpha(x)$  being the Bessel function.

It will be convenient to denote  $x^2 := (x_1^2, \dots, x_d^2)$ . Combining (2.1)-(2.3), we arrive at

$$(2.4) \quad \begin{aligned} P_n^{\alpha, \delta}(x^2, y^2) &= c_\alpha \int_{[0, \pi]^d} P_n^{\alpha, \delta}(z(x, y, \theta), 0) d\mu_{x, y}^\alpha(\theta) \\ &= c_\alpha \int_{[0, \pi]^d} L_n^{|\alpha| + \delta + d} \left( \|x\|_2^2 + \|y\|_2^2 + \sum_{i=1}^d x_i y_i \cos \theta_i \right) d\mu_{x, y}^\alpha(\theta), \end{aligned}$$

where  $c_\alpha = (2\pi)^{-d/2} 2^{|\alpha|} \prod_{i=1}^d \Gamma(\alpha_i + 1)$ ,  $z(x, y, \theta) = (z_1(x, y, \theta), \dots, z_d(x, y, \theta))$  with  $z_i(x, y, \theta) = x_i^2 + y_i^2 + 2x_i y_i \cos \theta_i$ , and

$$(2.5) \quad d\mu_{x, y}^\alpha(\theta) := e^{-\sum_{i=1}^d x_i y_i \cos \theta_i} \prod_{i=1}^d j_{\alpha_i - 1/2}(x_i y_i \sin \theta_i) \sin^{2\alpha_i} \theta_i d\theta.$$

Some standard asymptotic properties of Laguerre functions will be needed. The univariate Laguerre functions  $\mathcal{L}_n^\alpha$  are defined by

$$(2.6) \quad \mathcal{L}_n^\alpha(x) := \left( \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \right)^{1/2} e^{-x/2} x^{\alpha/2} L_n^\alpha(x).$$

**Lemma 2.1.** *Set  $N := 4n + 2\alpha + 2$ . The Laguerre functions  $\mathcal{L}_n^\alpha$  satisfy*

$$(2.7) \quad |\mathcal{L}_n^\alpha(x)| \leq c \begin{cases} (xN)^{\alpha/2}, & 0 < x \leq 1/N, \\ (xN)^{-1/4}, & 1/N \leq x \leq N/2, \\ N^{-1/4}(N^{1/3} + |N - x|)^{-1/4}, & N/2 \leq x \leq 3N/2, \\ e^{-\gamma x}, & x \geq 3N/2, \end{cases}$$

where  $\gamma > 0$  is an absolute constant.

This lemma is contained in [16, §8.22] (see also [17, Lemma 1.5.3]). Using that  $\Gamma(n + \alpha + 1)/\Gamma(n + 1) \sim n^\alpha$  one easily extracts from (2.7) the estimates

$$(2.8) \quad e^{-x/2} |\mathcal{L}_n^\alpha(x)| \leq cn^{\alpha/2 - 1/4} x^{-\alpha/2 - 1/4}, \quad x \in \mathbb{R}_+ \setminus (N/2, 3N/2),$$

and, for  $N/2 \leq x \leq 3N/2$ ,

$$(2.9) \quad e^{-x/2} |\mathcal{L}_n^\alpha(x)| \leq cx^{-\alpha/2} n^{\alpha/2 - 1/4} (n^{1/3} + |4n + 2\alpha + 2 - x|)^{-1/4}.$$

Also, from (2.7)

$$(2.10) \quad e^{-x/2} |\mathcal{L}_n^\alpha(x)| \leq cn^\alpha, \quad x \in \mathbb{R}_+,$$

and since  $\|\mathcal{L}_n^\alpha\|_\infty \leq c$ , again by (2.7),

$$(2.11) \quad e^{-x/2} |\mathcal{L}_n^\alpha(x)| \leq c(n/x)^{\alpha/2}, \quad x \in \mathbb{R}_+.$$

Let  $K_n^\alpha(x, y)$  be the reproducing kernel of the Laguerre polynomials. Then

$$(2.12) \quad K_n^\alpha(x, y) = c_\alpha \sum_{j=0}^n \frac{L_j^\alpha(x) L_j^\alpha(y)}{L_j^\alpha(0)}, \quad x, y \in \mathbb{R}_+.$$

The Christoffel function is defined by

$$(2.13) \quad \lambda_n^\alpha(x) := [K_n^\alpha(x, x)]^{-1}, \quad x \in \mathbb{R}_+.$$

For this function it is known that (see [11] and the references therein)

$$(2.14) \quad c_1 \varphi_n(x) \leq \frac{\lambda_n^\alpha(x)}{(x + \frac{1}{n})^\alpha e^{-x}} \leq c_2 \varphi_n(x), \quad 0 \leq x \leq 4n,$$

where

$$(2.15) \quad \varphi_n(x) := \sqrt{\frac{x + \frac{1}{n}}{4n - x + (4n)^{1/3}}}.$$

There are sharp estimates for  $L_n^\alpha(x)$  in terms of  $\varphi_n(x)$ . For any  $x > 0$ , let  $t_{k_x, n}$  denote the/a zero of  $L_n^\alpha(x)$  that is closest to  $x$ . Then (see e.g. [11])

$$(2.16) \quad [L_n^\alpha(x)]^2 \left(x + \frac{1}{n}\right)^{\alpha+1} e^{-x} \sim n^\alpha \varphi_n(x) \frac{(x - t_{k_x, n})^2}{(t_{k_x, n} - t_{k_x \pm 1, n})^2}, \quad x \in [t_{1, n}, t_{n, n}].$$

Here and in the following  $t_{1, n}, \dots, t_{n, n}$  denote the zeros of  $L_n^\alpha(x)$ . They are known to satisfy [16, §6.31]

$$(2.17) \quad cn^{-1} \leq t_{1, n} < t_{2, n} < \dots < t_{n, n} \leq 4n + 2\alpha + 2 - c(4n)^{1/3}.$$

Furthermore (see [16, (6.31.11)]),

$$(2.18) \quad c_* \frac{\nu^2}{n} \leq t_{\nu, n} \leq \frac{4\nu^2}{n} + c(\alpha) \frac{\nu}{n} \quad \text{and hence} \quad t_{\nu, n} \sim \frac{\nu^2}{n}.$$

In addition (see [11] and the references therein),

$$(2.19) \quad t_{\nu+1, n} - t_{\nu, n} \sim \varphi_n(t_{\nu, n}).$$

Therefore, if  $\nu \leq (1 - \varepsilon)n$  for some  $\varepsilon > 0$ , then by (2.18)  $t_{\nu, n} \leq (1 - \varepsilon)^2 4n + c(\alpha)$ , and hence, using (2.19) and (2.15),

$$(2.20) \quad t_{\nu+1, n} - t_{\nu, n} \sim \frac{\nu}{n} \quad \text{if} \quad \nu \leq (1 - \varepsilon)n.$$

On the other hand, by (2.19) and (2.15), in general,

$$(2.21) \quad \frac{c'}{n} \leq t_{\nu+1, n} - t_{\nu, n} \leq c'' n^{1/3}.$$

We will need the Gaussian quadrature formula with weight  $t^\alpha e^{-t}$  on  $(0, \infty)$  [16]:

$$(2.22) \quad \int_0^\infty f(t) t^\alpha e^{-t} dt \sim \sum_{\nu=1}^n w_{\nu, n} f(t_{\nu, n}), \quad w_{\nu, n} := \lambda_n^\alpha(t_{\nu, n}),$$

where  $t_{\nu, n}$  are the zeros of  $L_n^\alpha(t)$  and  $\lambda_n^\alpha(x)$  is the Christoffel function, defined in (2.13). This quadrature is exact for all algebraic polynomials of degree  $2n - 1$ .

### 3. LOCALIZED KERNELS ASSOCIATED WITH LAGUERRE FUNCTIONS

**3.1. The setting.** There are three kinds of univariate Laguerre functions considered in the literature (see [17]), defined by

$$(3.1) \quad \mathcal{F}_n^\alpha(x) := \left( \frac{2\Gamma(n+1)}{\Gamma(n+\alpha+1)} \right)^{1/2} e^{-x^2/2} L_n^\alpha(x^2),$$

$\mathcal{L}_n^\alpha(x)$  have already been defined in (2.6), and

$$(3.2) \quad \mathcal{M}_n^\alpha(x) := (2x)^{1/2} \mathcal{L}_n^\alpha(x^2).$$

It is well known that  $\{\mathcal{F}_n^\alpha\}_{n \geq 0}$  is an orthonormal basis for the weighed space  $L^2(\mathbb{R}_+, x^{2\alpha+1})$ , while  $\{\mathcal{L}_n^\alpha\}_{n \geq 0}$  and  $\{\mathcal{M}_n^\alpha\}_{n \geq 0}$  are orthonormal bases for  $L^2(\mathbb{R}_+)$ .

Throughout this paper we will use standard multi-index notation. Thus, for  $x \in \mathbb{R}^d$  and  $\alpha \in \mathbb{R}_+^d$ , we write  $x^\alpha := x_1^{\alpha_1} \dots x_d^{\alpha_d}$ . We will use  $\mathbf{1}$  to denote the vector

$\mathbf{1} := (1, 1, \dots, 1)$ . Then, for instance,  $x^{1/2} := x_1^{1/2} \dots x_d^{1/2}$ . The  $d$ -dimensional Laguerre functions are defined by

$$(3.3) \quad \mathcal{F}_\nu^\alpha(x) := \mathcal{F}_{\nu_1}^{\alpha_1}(x_1) \dots \mathcal{F}_{\nu_d}^{\alpha_d}(x_d),$$

$$(3.4) \quad \mathcal{L}_\nu^\alpha(x) := \mathcal{L}_{\nu_1}^{\alpha_1}(x_1) \dots \mathcal{L}_{\nu_d}^{\alpha_d}(x_d),$$

$$(3.5) \quad \mathcal{M}_\nu^\alpha(x) := \mathcal{M}_{\nu_1}^{\alpha_1}(x_1) \dots \mathcal{M}_{\nu_d}^{\alpha_d}(x_d),$$

where  $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}_0^d$  and  $\alpha = (\alpha_1, \dots, \alpha_d)$ . Clearly,  $x^{-\alpha} e^{|x|} \mathcal{L}_\nu^\alpha(x)$  is a polynomial of degree  $n = |\nu| = \nu_1 + \dots + \nu_d$  and

$$(3.6) \quad \mathcal{F}_\nu^\alpha(x) = 2^{d/2} x^{-\alpha} \mathcal{L}_\nu^\alpha(x_1^2, \dots, x_d^2).$$

Evidently,  $\{\mathcal{F}_\nu^\alpha\}$  is an orthonormal basis for the weighed space  $L^2(\mathbb{R}_+^d, w_\alpha)$ ,  $w_\alpha(x) := x^{2\alpha+1}$ , while  $\{\mathcal{L}_\nu^\alpha\}$  and  $\{\mathcal{M}_\nu^\alpha\}$  are orthonormal bases for  $L^2(\mathbb{R}_+^d)$  (with weight 1).

We will utilize the basis  $\{\mathcal{F}_\nu^\alpha\}$  to the construction of frames for the space  $L^2(w_\alpha) := L^2(\mathbb{R}_+^d, w_\alpha)$ . The same scheme based on  $\{\mathcal{L}_\nu^\alpha\}$  or  $\{\mathcal{M}_\nu^\alpha\}$  can be used for the construction of frames in  $L^2(\mathbb{R}_+^d)$ .

As explained in the introduction, kernels of type (1.3) will play a critical role in the present paper. For our purposes we will be considering cut-off functions  $\hat{a}$  that satisfy:

**Definition 3.1.** *A function  $\hat{a} \in C^\infty[0, \infty)$  is said to be admissible of type (a) or type (b) if  $\hat{a}$  satisfies one of the following conditions:*

- (a)  $\text{supp } \hat{a} \subset [0, 1 + v]$ ,  $\hat{a}(t) = 1$  on  $[0, 1]$ ,  $v > 0$ ; or
- (b)  $\text{supp } \hat{a} \subset [u, 1 + v]$ , where  $0 < u < 1$  and  $v > 0$ .

Here  $u, v$  are fixed constants.

For an admissible function  $\hat{a}$  we introduce the kernels

$$(3.7) \quad \Lambda_n(x, y) := \sum_{m=0}^{\infty} \hat{a}\left(\frac{m}{n}\right) \mathcal{F}_m^\alpha(x, y) \quad \text{with} \quad \mathcal{F}_m^\alpha(x, y) := \sum_{|\nu|=m} \mathcal{F}_\nu^\alpha(x) \mathcal{F}_\nu^\alpha(y),$$

$$(3.8) \quad \tilde{\Lambda}_n(x, y) := \sum_{m=0}^{\infty} \hat{a}\left(\frac{m}{n}\right) \mathcal{L}_m^\alpha(x, y) \quad \text{with} \quad \mathcal{L}_m^\alpha(x, y) := \sum_{|\nu|=m} \mathcal{L}_\nu^\alpha(x) \mathcal{L}_\nu^\alpha(y),$$

$$(3.9) \quad \Lambda_n^*(x, y) := \sum_{m=0}^{\infty} \hat{a}\left(\frac{m}{n}\right) \mathcal{M}_m^\alpha(x, y) \quad \text{with} \quad \mathcal{M}_m^\alpha(x, y) := \sum_{|\nu|=m} \mathcal{M}_\nu^\alpha(x) \mathcal{M}_\nu^\alpha(y).$$

The rapid decay of the kernels  $\Lambda_n(x, y)$ ,  $\tilde{\Lambda}_n(x, y)$ , and  $\Lambda_n^*(x, y)$  and their partial derivatives away from the main diagonal  $y = x$  in  $\mathbb{R}_+^d \times \mathbb{R}_+^d$  will be vital for our further development.

**3.2. The localization of  $\Lambda_n$  and its partial derivatives.** Recall the definition of the weight  $W_\alpha(n; x) := \prod_{i=1}^d (x_i + n^{-\frac{1}{2}})^{2\alpha_i+1}$ .

**Theorem 3.2.** *Let  $\hat{a}$  be admissible and let  $\sigma > 0$ . Then there is a constant  $c_\sigma$  depending only on  $\sigma$ ,  $\alpha$ , and  $\hat{a}$  such that for  $x, y \in \mathbb{R}_+^d$*

$$(3.10) \quad |\Lambda_n(x, y)| \leq c_\sigma \frac{n^{d/2}}{\sqrt{W_\alpha(n; x)} \sqrt{W_\alpha(n; y)} (1 + n^{1/2} \|x - y\|)^\sigma},$$

and furthermore, for  $1 \leq r \leq d$ ,

$$(3.11) \quad \left| \frac{\partial}{\partial x_r} \Lambda_n(x, y) \right| \leq c_\sigma \frac{n^{(d+1)/2}}{\sqrt{W_\alpha(n; x)} \sqrt{W_\alpha(n; y)} (1 + n^{1/2} \|x - y\|)^\sigma}.$$

Here the dependence of  $c_\sigma$  on  $\widehat{a}$  is of the form  $c_\sigma = c(\sigma, \alpha) \max_{0 \leq l \leq k} \|\widehat{a}^{(l)}\|_{L^\infty}$ , where  $k \geq \sigma + 2|\alpha| + d/2$ .

In addition to this, there exists a constant  $\varrho > 0$  such that if  $x, y \in \mathbb{R}_+^d$  and  $\max\{\|x\|, \|y\|\} \geq (6(1+v)n + 3\alpha + 3)^{1/2}$ , then

$$(3.12) \quad |\Lambda_n(x, y)| \leq c_\sigma \frac{e^{-\varrho \max\{\|x\|, \|y\|\}^2}}{(1 + n^{1/2} \|x - y\|)^\sigma}$$

and, for  $1 \leq r \leq d$ ,

$$(3.13) \quad \left| \frac{\partial}{\partial x_r} \Lambda_n(x, y) \right| \leq c_\sigma \frac{e^{-\varrho \max\{\|x\|, \|y\|\}^2}}{(1 + n^{1/2} \|x - y\|)^\sigma}.$$

To keep our exposition more fluid we relegate the proofs of these and the estimates to follow in this section to § 8.

We next use estimate (3.10) to bound the  $L^p$ -integral of  $\Lambda_n(x, y)$ , in particular, we show that  $\int_{\mathbb{R}_+^d} |\Lambda_n(x, y)| w_\alpha(y) dy \leq c < \infty$ .

**Proposition 3.3.** *For  $0 < p < \infty$ , we have*

$$(3.14) \quad \int_{\mathbb{R}_+^d} |\Lambda_n(x, y)|^p w_\alpha(y) dy \leq cn^{(d/2)(p-1)} W_\alpha(n; x)^{-(p-1)}, \quad x \in \mathbb{R}_+^d.$$

Estimate (3.14) is immediate from (3.10) and the following lemma which will be instrumental in the subsequent development.

**Lemma 3.4.** *If  $s \in \mathbb{R}$  and  $\sigma > d((2\|\alpha\| + 1)(|s| + 1) + 1)$ , then*

$$(3.15) \quad \int_{\mathbb{R}_+^d} \frac{w_\alpha(y) dy}{W_\alpha(n; y)^s (1 + n^{1/2} \|x - y\|)^\sigma} \leq \frac{cn^{-d/2}}{W_\alpha(n; x)^{s-1}}, \quad x \in \mathbb{R}_+^d.$$

We next give a lower bound estimate:

**Theorem 3.5.** *Let  $\widehat{a}$  be admissible in the sense of Definition 3.1 and  $|\widehat{a}| > c_\diamond > 0$  on  $[1, 1 + \tau]$ ,  $\tau > 0$ . Then for any  $\delta > 0$*

$$(3.16) \quad \int_{\mathbb{R}_+^d} |\Lambda_n(x, y)|^2 w_\alpha(y) dy \geq cn^{d/2} W_\alpha(n; x)^{-1} \quad x \in [0, \sqrt{(4 - \delta)n}]^d,$$

where  $c > 0$  depends only on  $\alpha$ ,  $d$ ,  $\tau$ ,  $\delta$ , and  $c_\diamond$ .

By the orthogonality of the Laguerre functions it readily follows that

$$\int_{\mathbb{R}_+^d} |\Lambda_n(x, y)|^2 w_\alpha(y) dy = \sum_{m=0}^{\infty} |\widehat{a}(m/n)|^2 \mathcal{F}_m^\alpha(x, x),$$

and hence Theorem 3.5 is an immediate consequence of the following lemma.

**Lemma 3.6.** *For any  $\varepsilon > 0$  and  $\delta > 0$  there exists a constant  $c > 0$  such that*

$$(3.17) \quad \sum_{m=n}^{n+[\delta\varepsilon n]} \mathcal{F}_m^\alpha(x, x) \geq cn^{d/2} W_\alpha(n; x)^{-1}, \quad x \in [0, \sqrt{(4 - \delta)n}]^d.$$



**3.3. The localization of  $\tilde{\Lambda}_n$  and its partial derivatives.** The localization of the kernels  $\tilde{\Lambda}_n$  can be deduced from the localization of  $\Lambda_n$  given above.

**Theorem 3.7.** *Let  $\hat{a}$  be admissible. Then for any  $\sigma > 0$  there is a constant  $c_\sigma > 0$  depending only on  $\sigma$ ,  $\alpha$ , and  $\hat{a}$  such that for  $x, y \in \mathbb{R}_+^d$ ,*

$$(3.18) \quad |\tilde{\Lambda}_n(x, y)| \leq c_\sigma \frac{n^{d/2}}{\prod_{i=1}^d (x_i + n^{-1})^{\frac{1}{4}} (y_i + n^{-1})^{\frac{1}{4}} (1 + n^{1/2} \|x^{1/2} - y^{1/2}\|)^\sigma},$$

and, for  $1 \leq r \leq d$ ,

$$(3.19) \quad \left| \frac{\partial}{\partial x_r} \tilde{\Lambda}_n(x, y) \right| \leq \frac{cn^{d/2+1}}{\prod_{i=1}^d (x_i + n^{-1})^{\frac{1}{4}} (y_i + n^{-1})^{\frac{1}{4}} (1 + n^{1/2} \|x^{1/2} - y^{1/2}\|)^\sigma}.$$

Here the dependence of  $c_\sigma$  on  $\hat{a}$  is as in Theorem 3.7.

Estimates for  $\tilde{\Lambda}_n$  like the ones of (3.12)-(3.16) can be extracted from (3.12)-(3.16). The results from this and the next subsections follow easily from Theorem 3.2, see §8.3.

**3.4. The localization of  $\Lambda_n^*$  and its partial derivatives.** The localization properties of  $\Lambda_n^*(x, y)$  appear simpler:

**Theorem 3.8.** *Let  $\hat{a}$  be admissible. Then for any  $\sigma > 0$  there is a constant  $c_\sigma$  such that for  $x, y \in \mathbb{R}_+^d$*

$$(3.20) \quad |\Lambda_n^*(x, y)| \leq c_\sigma \frac{n^{d/2}}{(1 + n^{1/2} \|x - y\|)^\sigma},$$

and, for  $1 \leq r \leq d$ ,

$$(3.21) \quad \left| \frac{\partial}{\partial x_r} \Lambda_n^*(x, y) \right| \leq c_\sigma \frac{n^{(d+1)/2}}{(1 + n^{1/2} \|x - y\|)^\sigma}.$$

Estimates for  $\Lambda_n^*$  similar to the ones of (3.12)-(3.16) can easily be obtained.

## 4. ADDITIONAL BACKGROUND MATERIAL

### 4.1. Norm equivalence.

**Proposition 4.1.** *Let  $0 < q \leq p \leq \infty$  and  $g \in V_n$  ( $n \geq 1$ ). Then*

$$(4.1) \quad \|g\|_p \leq cn^{(d+|\alpha|)(1/q-1/p)} \|g\|_q$$

and, for any  $s \in \mathbb{R}$ ,

$$(4.2) \quad \|W_\alpha(n; \cdot)^s g(\cdot)\|_p \leq cn^{(d/2)(1/q-1/p)} \|W_\alpha(n; \cdot)^{s+1/p-1/q} g(\cdot)\|_q.$$

Furthermore, for any  $s \in \mathbb{R}$

$$(4.3) \quad \|g\|_p \leq cn^M \|W_\alpha(n; \cdot)^s g(\cdot)\|_q,$$

where  $M$  depends only on  $\alpha, d, p, q$ , and  $s$ .

The proof of this proposition employs the localized kernels from §3 and is rather standard. For completeness we give it in §8.

**4.2. Maximal operator.** We define the “cube” centered at  $\xi \in \mathbb{R}_+^d$  of “radius”  $r > 0$  by  $Q_\xi(r) := \{x \in \mathbb{R}_+^d : \|x - \xi\| < r\}$ . Let  $\mathcal{M}_t$  be the maximal operator, defined by

$$(4.4) \quad \mathcal{M}_t f(x) := \sup_{Q: x \in Q} \left( \frac{1}{\mu(Q)} \int_Q |f(y)|^t w_\alpha(y) dy \right)^{1/t}, \quad x \in \mathbb{R}_+^d,$$

where the sup is over all “cubes”  $Q$  in  $\mathbb{R}_+^d$  with sides parallel to the coordinate axes which contain  $x$ . It is easy to see that

$$(4.5) \quad \mu(Q_\xi(r)) \sim r^d \prod_{j=1}^d (\xi_j + r)^{2\alpha_j + 1}.$$

Hence  $\mu(Q_\xi(2r)) \leq c\mu(Q_\xi(r))$ , i.e.  $\mu(\cdot)$  is a doubling measure. Therefore, the theory of maximal operators applies and the Fefferman-Stein vector-valued maximal inequality is valid (see [15]): If  $0 < p < \infty$ ,  $0 < q \leq \infty$ , and  $0 < t < \min\{p, q\}$ , then for any sequence of functions  $f_1, f_2, \dots$  on  $\mathbb{R}_+^d$

$$(4.6) \quad \left\| \left( \sum_{j=1}^{\infty} [\mathcal{M}_t f_j(\cdot)]^q \right)^{1/q} \right\|_p \leq c \left\| \left( \sum_{j=1}^{\infty} |f_j(\cdot)|^q \right)^{1/q} \right\|_p,$$

where  $c = c(p, q, t, d, \alpha)$ .

**4.3. Distributions on  $\mathbb{R}_+^d$ .** We will use as test functions the set  $\mathcal{S}_+$  of all functions  $\phi \in C^\infty([0, \infty)^d)$  such that

$$(4.7) \quad P_{\beta, \gamma}(\phi) := \sup_{x \in \mathbb{R}_+^d} |x^\gamma \partial^\beta \phi(x)| < \infty \quad \text{for all multi-indices } \gamma \text{ and } \beta,$$

with the topology on  $\mathcal{S}_+$  defined by the semi-norms  $P_{\beta, \gamma}$ . Then the space  $\mathcal{S}'_+$  of all temperate distributions on  $\mathbb{R}_+^d$  is defined as the set of all continuous linear functionals on  $\mathcal{S}_+$ . The pairing of  $f \in \mathcal{S}'_+$  and  $\phi \in \mathcal{S}_+$  will be denoted by  $\langle f, \phi \rangle := f(\bar{\phi})$  which is consistent with the inner product  $\langle f, g \rangle := \int_{\mathbb{R}_+^d} f(x) \bar{g}(x) w_\alpha(x) dx$  in  $L^2(\mathbb{R}_+^d, w_\alpha)$ .

It will be convenient for us to introduce the following “convolution”:

**Definition 4.2.** For functions  $\Phi : \mathbb{R}_+^d \times \mathbb{R}_+^d \rightarrow \mathbb{C}$  and  $f : \mathbb{R}_+^d \rightarrow \mathbb{C}$ , we define

$$(4.8) \quad \Phi * f(x) := \int_{\mathbb{R}_+^d} \Phi(x, y) f(y) w_\alpha(y) dy.$$

In general, if  $f \in \mathcal{S}'_+$  and  $\Phi : \mathbb{R}_+^d \times \mathbb{R}_+^d \rightarrow \mathbb{C}$  is such that  $\Phi(x, y)$  belongs to  $\mathcal{S}_+$  as a function of  $y$  ( $\Phi(x, \cdot) \in \mathcal{S}_+$ ), we define  $\Phi * f$  by

$$(4.9) \quad \Phi * f(x) := \langle f, \overline{\Phi(x, \cdot)} \rangle,$$

where on the right  $f$  acts on  $\overline{\Phi(x, y)}$  as a function of  $y$ .

We now give some properties of the above convolution that can be proved in a standard way.

**Lemma 4.3.** (a) If  $f \in \mathcal{S}'_+$  and  $\Phi(\cdot, \cdot) \in \mathcal{S}_+(\mathbb{R}_+^d \times \mathbb{R}_+^d)$ , then  $\Phi * f \in \mathcal{S}_+$ . Furthermore  $\mathcal{F}_n^\alpha * f \in V_n$ .

(b) If  $f \in \mathcal{S}'_+$ ,  $\Phi(\cdot, \cdot) \in \mathcal{S}_+(\mathbb{R}_+^d \times \mathbb{R}_+^d)$ , and  $\phi \in \mathcal{S}_+$ , then  $\langle \Phi * f, \phi \rangle = \langle f, \overline{\Phi * \phi} \rangle$ .

(c) If  $f \in \mathcal{S}'_+$ ,  $\Phi(\cdot, \cdot), \Psi(\cdot, \cdot) \in \mathcal{S}_+(\mathbb{R}_+^d \times \mathbb{R}_+^d)$ , and  $\Phi(y, x) = \Phi(x, y)$ ,  $\Psi(y, x) = \Psi(x, y)$ , then

$$(4.10) \quad \Psi * \bar{\Phi} * f(x) = \langle \Psi(x, \cdot), \Phi(\cdot, \cdot) \rangle * f.$$

Evidently the Laguerre functions  $\{\mathcal{F}_\nu^\alpha\}$  belong to  $\mathcal{S}_+$ . Moreover, the functions in  $\mathcal{S}_+$  can be characterized by the coefficients in their Laguerre expansions. Denote

$$(4.11) \quad P_r^*(\phi) := \sum_{n=0}^{\infty} (n+1)^r \|\mathcal{F}_n^\alpha * \phi\|_2 = \sum_{n=0}^{\infty} (n+1)^r \left( \sum_{|\nu|=n} |\langle \phi, \mathcal{F}_\nu^\alpha \rangle|^2 \right)^{1/2}.$$

**Lemma 4.4.** *A function  $\phi \in \mathcal{S}_+$  if and only if  $|\langle \phi, \mathcal{F}_\nu^\alpha \rangle| \leq c_k (|\nu| + 1)^{-k}$  for all multi-indices  $\nu$  and all  $k$ . Moreover, the topology in  $\mathcal{S}_+$  can be equivalently defined by the semi-norms  $P_r^*$ .*

The proof of this lemma is given in §8.

## 5. CONSTRUCTION OF FRAME ELEMENTS (NEEDLETS)

In this section we construct frames utilizing the localized kernels from §3 and a cubature formula on  $\mathbb{R}_+^d$ . As explained in the introduction, we will only use the Laguerre functions  $\{\mathcal{F}_\nu^\alpha\}$  defined in (3.3).

**5.1. Cubature formula.** We will utilize the Gaussian quadrature (2.22) for the construction of the needed cubature formula on  $\mathbb{R}_+^d$ . Given  $n \geq 1$ , we define, for  $\nu = 1, \dots, n$ ,

$$(5.1) \quad \xi_{\nu,n} := \sqrt{t_{\nu,n}} \quad \text{and} \quad c_{\nu,n} := \frac{1}{2} w_{\nu,n} e^{t_{\nu,n}} = \frac{1}{2} \lambda_n^\alpha(t_{\nu,n}) e^{t_{\nu,n}} = \frac{1}{2} \lambda_n^\alpha(\xi_{\nu,n}^2) e^{\xi_{\nu,n}^2},$$

where  $\{t_{\nu,n}\}$  are the zeros of  $L_n^\alpha(t)$  and  $\{w_{\nu,n}\}$  are the weights from (2.22).

It follows by (2.18) and (2.20)-(2.21) that

$$(5.2) \quad \xi_{\nu,n} \sim \frac{\nu}{\sqrt{n}},$$

$$(5.3) \quad \xi_{\nu+1,n} - \xi_{\nu,n} \sim n^{-1/2} \quad \text{if} \quad 1 \leq \nu \leq (1 - \varepsilon)n,$$

and, in general,

$$(5.4) \quad c_1 n^{-1/2} \leq \xi_{\nu+1,n} - \xi_{\nu,n} \leq c_2 n^{-1/6}.$$

Furthermore, using (2.14) and (2.19) we obtain

$$(5.5) \quad c_{\nu,n} \sim \varphi_n(t_{\nu,n}) t_{\nu,n}^\alpha \sim (t_{\nu+1,n} - t_{\nu,n}) t_{\nu,n}^\alpha \sim (\xi_{\nu+1,n} - \xi_{\nu,n}) \xi_{\nu,n}^{2\alpha+1}.$$

Now, for  $\gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{N}_0^d$  we set

$$(5.6) \quad c_{\gamma,n} := \prod_{j=1}^d c_{\gamma_j,n} \quad \text{and} \quad \xi_{\gamma,n} := (\xi_{\gamma_1,n}, \dots, \xi_{\gamma_d,n}).$$

**Proposition 5.1.** *The cubature formula*

$$(5.7) \quad \int_{\mathbb{R}_+^d} f(x) g(x) w_\alpha(x) dx \sim \sum_{\gamma_1=1}^n \cdots \sum_{\gamma_d=1}^n c_{\gamma,n} f(\xi_{\gamma,n}) g(\xi_{\gamma,n})$$

is exact for all  $f \in V_\ell$  and  $g \in V_m$  provided  $\ell + m \leq 2n - 1$ .

**Proof.** Evidently, it suffices to consider only the case  $d = 1$ . Suppose  $f \in V_\ell$  and  $g \in V_m$  with  $\ell + m \leq 2n - 1$ . Let  $f(x) =: F(x^2)e^{-x^2/2}$  and  $g(x) =: G(x^2)e^{-x^2/2}$ , where  $F \in \Pi_\ell^1$ ,  $G \in \Pi_m^1$  with  $\Pi_j^1$  being the set of all univariate polynomials of degree  $\leq j$ . Then using the properties of quadrature formula (2.22), we get

$$\begin{aligned} \int_0^\infty f(x)g(x)w_\alpha(x)dx &= \int_0^\infty F(x^2)G(x^2)x^{2\alpha+1}e^{-x^2}dx = \frac{1}{2} \int_0^\infty F(t)G(t)t^\alpha e^{-t}dt \\ &= \frac{1}{2} \sum_{\nu=1}^n w_{\nu,n}F(t_{\nu,n})G(t_{\nu,n}) = \sum_{\nu=1}^n \frac{1}{2} w_{\nu,n}F(\xi_{\nu,n}^2)G(\xi_{\nu,n}^2) \\ &= \sum_{\nu=1}^n \frac{1}{2} \lambda_n^\alpha(\xi_{\nu,n}^2) e^{\xi_{\nu,n}^2} f(\xi_{\nu,n})g(\xi_{\nu,n}), \end{aligned}$$

which completes the proof.  $\square$

To construct our frame elements we need the cubature formula from (5.7) with

$$(5.8) \quad n = n_j := \lfloor c_*^{-1}(1 + 11\delta)\sqrt{6} \cdot 4^j \rfloor + 1 \sim 4^j,$$

where  $0 < c_* \leq 1$  is the constant from (2.18) and  $0 < \delta < 1/26$  is an arbitrary but fixed constant. For  $j \geq 0$ , we define

$$(5.9) \quad \mathcal{X}_j := \{\xi \in \mathbb{R}_+^d : \xi = \xi_{\gamma,n_j}, 1 \leq \gamma_\ell \leq n_j, 1 \leq \ell \leq d\}.$$

Note that  $\#\mathcal{X}_j = n_j^d \sim 4^{jd}$ . Now, if  $\xi \in \mathcal{X}_j$  and  $\xi = \xi_{\gamma,n_j}$ , we set  $c_\xi := c_{\gamma,n_j}$ .

As an immediate consequence of Proposition 5.1 we get

**Corollary 5.2.** *The cubature formula*

$$(5.10) \quad \int_{\mathbb{R}_+^d} f(x)g(x)w_\alpha(x)dx \sim \sum_{\xi \in \mathcal{X}_j} c_\xi f(\xi)g(\xi)$$

is exact for all  $f \in V_\ell$  and  $g \in V_m$  provided  $\ell + m \leq 2n_j - 1$ .

**Tiling.** We next introduce rectangular tiles  $\{R_\xi\}$  with ‘‘centers’’ at the points  $\xi \in \mathcal{X}_j$ . Set  $I_1 := [0, (\xi_1 + \xi_2)/2]$  and

$$I_\nu := [(\xi_{\nu-1} + \xi_\nu)/2, (\xi_\nu + \xi_{\nu+1})/2], \quad \nu = 2, \dots, n_j,$$

where  $\xi_\nu := \xi_{\nu,n_j}$ ,  $\nu = 1, \dots, n_j$ , are from (5.1) and  $\xi_{n_j+1} := \xi_{n_j} + 2^{j/3}$ .

To every  $\xi = \xi_\gamma = (\xi_{\gamma_1}, \dots, \xi_{\gamma_d})$  in  $\mathcal{X}_j$  we associate a tile  $R_\xi$  defined by

$$(5.11) \quad R_\xi := I_{\gamma_1} \times \dots \times I_{\gamma_d}.$$

We also set

$$(5.12) \quad Q_j := \cup_{\xi \in \mathcal{X}_j} R_\xi.$$

Evidently, different tiles  $R_\xi$  do not overlap and  $Q_j \sim [0, 2^j]^d$ .

By (5.5) it readily follows that

$$(5.13) \quad c_\xi \sim \mu(R_\xi) := \int_{R_\xi} w_\alpha(x)dx \sim |R_\xi|w_\alpha(\xi) \sim |R_\xi|W_\alpha(4^j; \xi).$$

Assume  $\xi \in \mathcal{X}_j$ ,  $\xi := \xi_\gamma$ , and  $\|\xi\| \leq (1 + 4\delta)\sqrt{6} \cdot 2^j$ . By (2.18)  $\|\xi_\gamma\| \geq c_*^{1/2}\|\gamma\|n_j^{-1/2}$  and hence  $\|\gamma\| \leq c_*^{-1/2}(1 + 4\delta)\sqrt{6} \cdot 2^j n_j^{1/2} \leq (1 - \delta)n_j$ , where the last inequality follows by the selection of  $n_j$  in (5.8). Therefore, for  $\xi \in \mathcal{X}_j$

$$(5.14) \quad R_\xi \sim \xi + [-2^{-j}, 2^{-j}]^d \quad \text{and} \quad \mu(R_\xi) \sim 2^{-jd}w_\alpha(\xi) \quad \text{if} \quad \|\xi\| \leq (1 + 4\delta)\sqrt{6} \cdot 2^j,$$

while in general, for some positive constants  $c_1, c_2, c', c''$ ,

$$(5.15) \quad \xi + [-c_1 2^{-j}, c_1 2^{-j}]^d \subset R_\xi \subset \xi + [-c_2 2^{-j/3}, c_2 2^{-j/3}]^d \quad \text{and}$$

$$(5.16) \quad c' 2^{-jd} w_\alpha(\xi) \leq \mu(R_\xi) \leq c'' 2^{-jd/3} w_\alpha(\xi).$$

The following simple inequality is immediate from the definition of  $W_\alpha(n; x)$  in (1.2) and will be useful in what follows:

$$(5.17) \quad W_\alpha(4^j; y) \leq W_\alpha(4^j; x) (1 + 2^j \|x - y\|)^{2|\alpha|+d}, \quad x, y \in \mathbb{R}_+^d.$$

**5.2. Definition of Needlets.** Let  $\widehat{a}, \widehat{b}$  satisfy the conditions:

$$(5.18) \quad \widehat{a}, \widehat{b} \in C^\infty(\mathbb{R}), \quad \text{supp } \widehat{a}, \text{supp } \widehat{b} \subset [1/4, 4],$$

$$(5.19) \quad |\widehat{a}(t)|, |\widehat{b}(t)| > c > 0 \quad \text{if } t \in [1/3, 3],$$

$$(5.20) \quad \overline{\widehat{a}(t)} \widehat{b}(t) + \overline{\widehat{a}(4t)} \widehat{b}(4t) = 1 \quad \text{if } t \in [1/4, 1].$$

Hence,

$$(5.21) \quad \sum_{m=0}^{\infty} \overline{\widehat{a}(4^{-m}t)} \widehat{b}(4^{-m}t) = 1, \quad t \in [1, \infty).$$

It is readily seen that (e.g. [6]) for any  $\widehat{a}$  satisfying (5.18)-(5.19) there exists  $\widehat{b}$  satisfying (5.18)-(5.19) such that (5.20) holds.

Let  $\widehat{a}, \widehat{b}$  satisfy (5.18)-(5.20). Then we set

$$(5.22) \quad \Phi_0(x, y) := \mathcal{F}_0^\alpha(x, y), \quad \Phi_j(x, y) := \sum_{m=0}^{\infty} \widehat{a}\left(\frac{m}{4^{j-1}}\right) \mathcal{F}_m^\alpha(x, y), \quad \text{and}$$

$$(5.23) \quad \Psi_0(x, y) := \mathcal{F}_0^\alpha(x, y), \quad \Psi_j(x, y) := \sum_{m=0}^{\infty} \widehat{b}\left(\frac{m}{4^{j-1}}\right) \mathcal{F}_m^\alpha(x, y), \quad j \geq 1.$$

Let  $\mathcal{X}_j$  be the set defined in (5.9) and let  $c_\xi$  be the coefficients of cubature formula (5.10). We define the  $j$ th level *needlets* by

$$(5.24) \quad \varphi_\xi(x) := c_\xi^{1/2} \Phi_j(x, \xi) \quad \text{and} \quad \psi_\xi(x) := c_\xi^{1/2} \Psi_j(x, \xi), \quad \xi \in \mathcal{X}_j.$$

Set  $\mathcal{X} := \cup_{j=0}^{\infty} \mathcal{X}_j$ . We will use  $\mathcal{X}$  as an index set for our needlet systems  $\Phi$  and  $\Psi$ . For this reason, (possibly) identical points from different levels  $\mathcal{X}_j$  are considered as distinct elements of  $\mathcal{X}$ . We define

$$(5.25) \quad \Phi := \{\varphi_\xi\}_{\xi \in \mathcal{X}}, \quad \Psi := \{\psi_\xi\}_{\xi \in \mathcal{X}}.$$

We will term  $\{\varphi_\xi\}$  *analysis needlets* and  $\{\psi_\xi\}$  *synthesis needlets*.

**Localization of Needlets.** An immediate consequence of Theorem 3.2 is the estimate: For any  $\sigma > 0$  there exists a constant  $c_\sigma > 0$  such that for all  $x, y \in \mathbb{R}_+^d$

$$(5.26) \quad |\Phi_j(x, y)|, |\Psi_j(x, y)| \leq \frac{c_\sigma 2^{jd}}{\sqrt{W_\alpha(4^j, x)} \sqrt{W_\alpha(4^j, y)} (1 + 2^j \|x - y\|)^\sigma},$$

while  $c_\sigma 2^{jd}$  can be replaced by  $c(\sigma, L)2^{-jL}$  if  $\max\{\|x\|, \|y\|\} \geq (1+\delta)\sqrt{6} \cdot 2^j$ , where  $L > 0$  is an arbitrary constant but the constant  $c(\sigma, L)$  depends on  $L$  as well. We employ (5.26) and (5.13) to obtain for  $\xi \in \mathcal{X}_j$

$$(5.27) \quad |\varphi_\xi(x)|, |\psi_\xi(x)| \leq \frac{c2^{jd/2}}{\sqrt{W_\alpha(4^j, x)}(1+2^j\|x-\xi\|)^\sigma}, \quad x \in \mathbb{R}_+^d,$$

and

$$(5.28) \quad |\varphi_\xi(x)|, |\psi_\xi(x)| \leq \frac{c2^{-jL}}{\sqrt{W_\alpha(4^j, x)}(1+2^j\|x-\xi\|)^\sigma}, \quad \text{if } \|\xi\| \geq (1+\delta)\sqrt{6} \cdot 2^j.$$

We next show that  $\mathcal{S}'_+$  and  $L^p(\mathbb{R}_+^d)$  have discrete decompositions via needlets.

**Proposition 5.3.** (a) *If  $f \in \mathcal{S}'_+$ , then*

$$(5.29) \quad f = \sum_{j=0}^{\infty} \Psi_j * \bar{\Phi}_j * f \quad \text{in } \mathcal{S}'_+ \text{ and}$$

$$(5.30) \quad f = \sum_{\xi \in \mathcal{X}} \langle f, \varphi_\xi \rangle \psi_\xi \quad \text{in } \mathcal{S}'_+.$$

(b) *If  $f \in L^p(w_\alpha)$ ,  $1 \leq p < \infty$ , then (5.29) – (5.30) hold in  $L^p(w_\alpha)$ . Moreover, if  $1 < p < \infty$ , then the convergence in (5.29) – (5.30) is unconditional.*

**Proof.** (a) Note that  $\Psi_j * \bar{\Phi}_j(x, y)$  is well defined since  $\Psi_j(x, y)$  and  $\bar{\Phi}_j(x, y)$  are symmetric functions (e.g.  $\Psi_j(y, x) = \Psi_j(x, y)$ ). By (5.22)-(5.23) it follows that  $\Psi_0 * \bar{\Phi}_0 = \mathcal{F}_0^\alpha$  and

$$(5.31) \quad \Psi_j * \bar{\Phi}_j(x, y) = \sum_{m=4^{j-2}}^{4^j} \widehat{a}\left(\frac{m}{4^{j-1}}\right) \widehat{b}\left(\frac{m}{4^{j-1}}\right) \mathcal{F}_m^\alpha(x, y), \quad j \geq 1.$$

Hence, (5.21) and Lemma 4.4 imply (5.29). Evidently,  $\Psi_j(x, \cdot)$  and  $\overline{\bar{\Phi}_j(y, \cdot)}$  belong to  $V_{4^j}$  and using the cubature formula from Corollary 5.2, we infer

$$\begin{aligned} \Psi_j * \bar{\Phi}_j(x, y) &= \int_{\mathbb{R}_+^d} \Psi_j(x, u) \overline{\bar{\Phi}_j(y, u)} du \\ &= \sum_{\xi \in \mathcal{X}_j} c_\xi \Psi_j(x, \xi) \overline{\bar{\Phi}_j(y, \xi)} = \sum_{\xi \in \mathcal{X}_j} \psi_\xi(x) \overline{\varphi_\xi(y)}. \end{aligned}$$

Therefore,  $\Psi_j * \bar{\Phi}_j * f = \sum_{\xi \in \mathcal{X}_j} \langle f, \varphi_\xi \rangle \psi_\xi$  and combining this with (5.29) gives (5.30).

(b) In  $L^p$  identity (5.29) follows easily by the rapid decay of the kernels of the  $n$ th partial sums. We skip the details. In  $L^p$ , identity (5.30) follows as above. The unconditional convergence in  $L^p(w_\alpha)$ ,  $1 < p < \infty$ , is a consequence of Proposition 6.3 and Theorem 6.7 below.  $\square$

**Remark 5.4.** *Suppose that in the needlet construction  $\widehat{b} = \widehat{a}$  and  $\widehat{a} \geq 0$ . Then  $\varphi_\xi = \psi_\xi$  and (5.30) becomes  $f = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \psi_\xi$ . It is easily seen that this representation holds in  $L^2$  and  $\|f\|_{L^2} = \left( \sum_{\xi \in \mathcal{X}} |\langle f, \psi_\xi \rangle|^2 \right)^{1/2}$ ,  $f \in L^2$ , i.e.  $\{\psi_\xi\}_{\xi \in \mathcal{X}}$  is a tight frame for  $L^2(\mathbb{R}_+^d, w_\alpha)$ .*

## 6. LAGUERRE-TRIEBEL-LIZORKIN SPACES

We follow the general idea of using spectral decompositions (see e.g. [13], [18]) to introduce Triebel-Lizorkin spaces on  $\mathbb{R}_+^d$  in the context of Laguerre expansions. Our main goal is to show that these spaces can be characterized via needlet representations.

**6.1. Definition of Laguerre-Triebel-Lizorkin spaces.** Let a sequence of kernels  $\{\Phi_j\}$  be defined by

$$(6.1) \quad \Phi_0(x, y) := \mathcal{F}_0^\alpha(x, y) \quad \text{and} \quad \Phi_j(x, y) := \sum_{m=0}^{\infty} \widehat{a}\left(\frac{m}{4^{j-1}}\right) \mathcal{F}_m^\alpha(x, y), \quad j \geq 1,$$

where  $\{\mathcal{F}_m^\alpha(x, y)\}$  are from (3.7) and  $\widehat{a}$  obeys the conditions

$$(6.2) \quad \widehat{a} \in C^\infty[0, \infty), \quad \text{supp } \widehat{a} \subset [1/4, 4],$$

$$(6.3) \quad |\widehat{a}(t)| > c > 0, \quad \text{if } t \in [1/3, 3].$$

**Definition 6.1.** Let  $s, \rho \in \mathbb{R}$ ,  $0 < p < \infty$ , and  $0 < q \leq \infty$ . Then the Laguerre-Triebel-Lizorkin space  $F_{pq}^{s\rho} := F_{pq}^{s\rho}(\mathcal{F}^\alpha)$  is defined as the set of all distributions  $f \in \mathcal{S}'_+$  such that

$$(6.4) \quad \|f\|_{F_{pq}^{s\rho}} := \left\| \left( \sum_{j=0}^{\infty} \left[ 2^{sj} W_\alpha(4^j; \cdot)^{-\rho/d} |\Phi_j * f(\cdot)| \right]^q \right)^{1/q} \right\|_p < \infty$$

with the usual modification when  $q = \infty$ .

As is shown in Theorem 6.7 below the above definition is independent of the choice of  $\widehat{a}$  as long as  $\widehat{a}$  satisfies (6.2)-(6.3).

**Proposition 6.2.** For all  $s, \rho \in \mathbb{R}$ ,  $0 < p < \infty$ , and  $0 < q \leq \infty$ ,  $F_{pq}^{s\rho}$  is a (quasi-)Banach space which is continuously embedded in  $\mathcal{S}'_+$ .

**Proof.** The completeness of the space  $F_{pq}^{s\rho}$  follows easily (see e.g. [18], p. 49) by the continuous embedding of  $F_{pq}^{s\rho}$  in  $\mathcal{S}'_+$ , which we establish next.

Let  $\{\Phi_j\}$  be the kernels from the definition of  $F_{pq}^{s\rho}$  with  $\widehat{a}$  obeying (6.2)-(6.3) that are the same as (5.18)-(5.19). As already indicated there exists a function  $\widehat{b}$  satisfying (5.18)-(5.20). We use this function to define  $\{\Psi_j\}$  as in (5.23). Assume  $f \in F_{pq}^{s\rho}$ . Then by Proposition 5.3  $f = \sum_{j=0}^{\infty} \Psi_j * \Phi_j * f$  in  $\mathcal{S}'_+$  and hence

$$\langle f, \phi \rangle = \sum_{j=0}^{\infty} \langle \Psi_j * \overline{\Phi}_j * f, \phi \rangle, \quad \phi \in \mathcal{S}_+.$$

We now employ (5.31) and the Cauchy-Schwarz inequality to obtain, for  $j \geq 2$ ,

$$\begin{aligned} |\langle \Psi_j * \overline{\Phi}_j * f, \phi \rangle|^2 &= \left| \sum_{m=4^{j-2}+1}^{4^j} \overline{\widehat{a}\left(\frac{m}{4^{j-1}}\right) \widehat{b}\left(\frac{m}{4^{j-1}}\right)} \langle \mathcal{F}_m^\alpha * f, \mathcal{F}_m^\alpha * \phi \rangle \right|^2 \\ &\leq \sum_{m=4^{j-2}+1}^{4^j} \left| \widehat{a}\left(\frac{m}{4^{j-1}}\right) \right|^2 \|\mathcal{F}_m^\alpha * f\|_2^2 \sum_{m=4^{j-2}+1}^{4^j} \left| \widehat{b}\left(\frac{m}{4^{j-1}}\right) \right|^2 \|\mathcal{F}_m^\alpha * \phi\|_2^2 \\ &\leq \|\Phi_j * f\|_2^2 \sum_{m=4^{j-2}+1}^{4^j} \|\mathcal{F}_m^\alpha * \phi\|_2^2. \end{aligned}$$

Using inequality (4.3) we get

$$\|\Phi_j * f\|_2 \leq c2^{j(M+|s|)} \|2^{sj} W_\alpha(2^j; \cdot)^{-\rho/d} \Phi_j * f(\cdot)\|_p \leq c2^{j(M+|s|)} \|f\|_{F_{pq}^{s\rho}},$$

where  $M$  depends on  $p, \alpha, d$ , and  $\rho$ . From the above estimates we infer

$$|\langle \Psi_j * \bar{\Phi}_j * f, \phi \rangle| \leq c2^{-j} \|f\|_{F_{pq}^{s\rho}} 2^{jk} \sum_{4^{j-2} < m \leq 4^j} \|\mathcal{F}_m^\alpha * f\|_2 \leq c2^{-j} \|f\|_{F_{pq}^{s\rho}} P_k^*(\phi)$$

for  $k \geq M + |s| + 1$ . A similar estimate trivially holds for  $j = 0, 1$ . Summing up we get

$$|\langle f, \phi \rangle| \leq c \|f\|_{F_{pq}^{s\rho}} P_k^*(\phi),$$

which completes the proof.  $\square$

**Proposition 6.3.** *The following identification holds:*

$$(6.5) \quad F_{p2}^{00} \sim L^p(w_\alpha), \quad 1 < p < \infty,$$

with equivalent norms.

The proof of this proposition is the same as the proof of Proposition 4.3 in [12] in the case of spherical harmonics. We omit it. Almost arbitrary  $L^p$  multipliers for Laguerre expansions can be used for the proof. However, since we cannot find in the literature any multipliers for the Laguerre expansions we use in the present paper, we next give easy to prove but non-optimal multipliers.

**Proposition 6.4.** *Let  $k$  be sufficiently large integer ( $k > (5/2)|\alpha| + (7/4)d + 3$  will do) and suppose  $m \in C^k(\mathbb{R}_+)$  obeys*

$$(6.6) \quad \sup_{t \in \mathbb{R}_+} |t^j m^{(j)}(t)| \leq c \quad \text{for } j = 0, 1, \dots, k.$$

Then the operator  $T_m^\alpha(f) := \sum_{j=0}^\infty m(j) \mathcal{F}_j^\alpha * f$  is bounded on  $L^p(w_\alpha)$ ,  $1 < p < \infty$ .

The proof is given in §9.

**6.2. Needlet Decomposition of Laguerre-Triebel-Lizorkin Spaces.** As a companion to  $F_{pq}^{s\rho}$  we now introduce the sequence spaces  $f_{pq}^{s\rho}$ . Here  $\{\mathcal{X}_j\}_{j=0}^\infty$  is the sequence of points from (5.9) with associated tiles  $\{R_\xi\}_{\xi \in \mathcal{X}_j}$ , defined in (5.11). Just as in the definition of needlets in §5, we set  $\mathcal{X} := \cup_{j \geq 0} \mathcal{X}_j$ .

**Definition 6.5.** *Suppose  $s, \rho \in \mathbb{R}$ ,  $0 < p < \infty$ , and  $0 < q \leq \infty$ . Then  $f_{pq}^{s\rho}$  is defined as the space of all complex-valued sequences  $h := \{h_\xi\}_{\xi \in \mathcal{X}}$  such that*

$$(6.7) \quad \|h\|_{f_{pq}^{s\rho}} := \left\| \left( \sum_{j=0}^\infty 2^{sjq} \sum_{\xi \in \mathcal{X}_j} [|h_\xi| W_\alpha(4^j; \xi)^{-\rho/d} \tilde{\mathbb{1}}_{R_\xi}(\cdot)]^q \right)^{1/q} \right\|_p < \infty$$

with the usual modification for  $q = \infty$ . Recall that  $\tilde{\mathbb{1}}_{R_\xi} := \mu(R_\xi)^{-1/2} \mathbb{1}_{R_\xi}$ .

In analogy to the classical case on  $\mathbb{R}^d$  we introduce “analysis” and “synthesis” operators by

$$(6.8) \quad S_\varphi : f \rightarrow \{\langle f, \varphi_\xi \rangle\}_{\xi \in \mathcal{X}} \quad \text{and} \quad T_\psi : \{h_\xi\}_{\xi \in \mathcal{X}} \rightarrow \sum_{\xi \in \mathcal{X}} h_\xi \psi_\xi.$$

We next show that the operator  $T_\psi$  is well defined on  $f_{pq}^{s\rho}$ .



**Lemma 6.6.** *Let  $s, \rho \in \mathbb{R}$ ,  $0 < p < \infty$ , and  $0 < q \leq \infty$ . Then for any  $h \in f_{pq}^{s\rho}$ ,  $T_\psi h := \sum_{\xi \in \mathcal{X}} h_\xi \psi_\xi$  converges in  $\mathcal{S}'_+$ . Moreover, the operator  $T_\psi : f_{pq}^{s\rho} \rightarrow \mathcal{S}'_+$  is continuous, i.e. there exist constants  $k > 0$  and  $c > 0$  such that*

$$(6.9) \quad |\langle T_\psi h, \phi \rangle| \leq c P_k^*(\phi) \|h\|_{f_{pq}^{s\rho}} \quad \text{for } h \in f_{pq}^{s\rho} \text{ and } \phi \in \mathcal{S}_+.$$

**Proof.** Let  $h \in f_{pq}^{s\rho}$ . Using the definition of  $f_{pq}^{s\rho}$  we obtain

$$2^{js} |h_\xi| W_\alpha(4^j; \xi)^{-\rho/d} \|\tilde{\mathbb{1}}_{R_\xi}(\cdot)\|_p \leq \|h\|_{f_{pq}^{s\rho}} \quad \text{for } \xi \in \mathcal{X}_j, j \geq 0.$$

But (5.16) gives  $\|\tilde{\mathbb{1}}_{R_\xi}\|_p = \mu(R_\xi)^{1/p-1/2} \geq c[2^{-jd} W_\alpha(4^j, \xi)]^{1/p-1/2}$  for  $\xi \in \mathcal{X}_j$  and since  $2^{-j(2|\alpha|+d)} \leq W_\alpha(4^j, \xi) \leq c2^{j(2|\alpha|+d)}$  it follows that for  $\xi \in \mathcal{X}_j$

$$(6.10) \quad |h_\xi| \leq c2^{jM} \|h\|_{f_{pq}^{s\rho}} \quad \text{with } M := |s| + 2(|\alpha| + d)(|\rho|/d + |1/p - 1/2|).$$

By Lemma 4.4  $\phi = \sum_{n=0}^{\infty} \mathcal{F}_n^\alpha * \phi$  in  $\mathcal{S}_+$  for  $\phi \in \mathcal{S}_+$  and hence for  $\xi \in \mathcal{X}_j$

$$\psi_\xi(x) := c_\xi^{1/2} \Psi_j(x, \xi) = c_\xi^{1/2} \sum_{4^{j-2} < m < 4^j} \hat{b}\left(\frac{m}{4^{j-1}}\right) \mathcal{F}_m^\alpha(x, \xi), \quad c_\xi \sim |R_\xi| W_\alpha(4^j, \xi).$$

Therefore,

$$\langle \psi_\xi, \phi \rangle = c_\xi^{1/2} \sum_{4^{j-2} < m < 4^j} \hat{b}\left(\frac{m}{4^{j-1}}\right) \mathcal{F}_m^\alpha * \bar{\phi}$$

and hence

$$|\langle \psi_\xi, \phi \rangle| \leq c2^{-j(|\alpha|+d)} \sum_{4^{j-2} < m < 4^j} \|\mathcal{F}_m^\alpha * \phi\|_\infty.$$

Since  $\mathcal{F}_m^\alpha * \phi \in V_m$ , by Proposition 4.1  $\|\mathcal{F}_m^\alpha * \phi\|_\infty \leq cm^{(d+|\alpha|)/2} \|\mathcal{F}_m^\alpha * \phi\|_2$  and hence

$$|\langle \psi_\xi, \phi \rangle| \leq c2^{j(2|\alpha|+2d)} \sum_{4^{j-2} < m < 4^j} \|\mathcal{F}_m^\alpha * \phi\|_2.$$

This along with (6.10) and the fact that  $\#\mathcal{X}_j \leq c4^{jd}$  yields, for  $\phi \in \mathcal{S}_+$ ,

$$(6.11) \quad \begin{aligned} \sum_{\xi \in \mathcal{X}} |h_\xi| |\langle \psi_\xi, \phi \rangle| &\leq \sum_{j=0}^{\infty} \sum_{\xi \in \mathcal{X}_j} |h_\xi| |\langle \psi_\xi, \phi \rangle| \\ &\leq c \|h\|_{f_{pq}^{s\rho}} \sum_{j=0}^{\infty} (\#\mathcal{X}_j) 2^{j(M+2|\alpha|+2d)} \sum_{4^{j-2} < m < 4^j} \|\mathcal{F}_m^\alpha * \phi\|_2 \\ &\leq c \|h\|_{f_{pq}^{s\rho}} \sum_{m=0}^{\infty} (m+1)^k \|\mathcal{F}_m^\alpha * \phi\|_2 \sum_{j=0}^{\infty} 2^{j(M+2|\alpha|+4d+1-k)} \\ &\leq c \|h\|_{f_{pq}^{s\rho}} P_k^*(\phi), \end{aligned}$$

where  $k := \lfloor M + 2|\alpha| + 4d + 2 \rfloor > M + 2|\alpha| + 4d + 1$ . Therefore, the series  $\sum_{\xi \in \mathcal{X}} h_\xi \psi_\xi$  converges in  $\mathcal{S}'$ . We define  $T_\psi h$  by  $\langle T_\psi h, \phi \rangle := \sum_{\xi \in \mathcal{X}} h_\xi \langle \psi_\xi, \phi \rangle$  for all  $\phi \in \mathcal{S}$ . Estimate (6.9) follows by (6.11).  $\square$

We now present our main result on Laguerre-Triebel-Lizorkin spaces.

**Theorem 6.7.** *Let  $s, \rho \in \mathbb{R}$ ,  $0 < p < \infty$  and  $0 < q \leq \infty$ . Then the operators  $S_\varphi : F_{pq}^{s\rho} \rightarrow f_{pq}^{s\rho}$  and  $T_\psi : f_{pq}^{s\rho} \rightarrow F_{pq}^{s\rho}$  are bounded and  $T_\psi \circ S_\varphi = \text{Id}$  on  $F_{pq}^{s\rho}$ . Consequently,  $f \in F_{pq}^{s\rho}$  if and only if  $\{\langle f, \varphi_\xi \rangle\}_{\xi \in \mathcal{X}} \in f_{pq}^{s\rho}$  and*

$$(6.12) \quad \|f\|_{F_{pq}^{s\rho}} \sim \|\{\langle f, \varphi_\xi \rangle\}\|_{f_{pq}^{s\rho}}.$$

In addition, the definition of  $F_{pq}^{sp}$  is independent of the particular selection of  $\widehat{a}$  satisfying (6.2)–(6.3).

To prove this theorem we need several lemmas with proofs given in Section 9. Assume that  $\{\Phi_j\}$  are the kernels from the definition of Laguerre-Triebel-Lizorkin spaces and  $\{\varphi_\xi\}_{\xi \in \mathcal{X}}$  and  $\{\psi_\xi\}_{\xi \in \mathcal{X}}$  are needlet systems defined as in (5.24) with no connection between the functions  $\widehat{a}$ 's from (6.1) and (5.22). We also assume that  $p, q$  from the hypothesis of Theorem 6.7 are fixed and we choose  $0 < t < \min\{p, q\}$ .

**Lemma 6.8.** *For any  $\sigma > d$  there exists a constant  $c_\sigma > 0$  such that*

$$(6.13) \quad |\Phi_j * \psi_\xi(x)| \leq \frac{c_\sigma}{\mu(R_\xi)^{1/2}(1+2^m\|x-\xi\|)^\sigma}, \quad \xi \in \mathcal{X}_m, \quad j-1 \leq m \leq j+1,$$

and  $\Phi_j * \psi_\xi \equiv 0$  for  $\xi \in \mathcal{X}_m$  if  $|m-j| \geq 2$ , were  $\mathcal{X}_m := \emptyset$  if  $m < 0$ .

**Definition 6.9.** *For any collection of complex numbers  $\{h_\xi\}_{\xi \in \mathcal{X}_j}$  ( $j \geq 0$ ), we define*

$$(6.14) \quad h_j^*(x) := \sum_{\eta \in \mathcal{X}_j} \frac{|h_\eta|}{(1+2^j\|\eta-x\|)^\lambda}$$

and

$$(6.15) \quad h_\xi^* := h_j^*(\xi), \quad \xi \in \mathcal{X}_j,$$

where  $\lambda := 2d + 2(|\alpha| + 3d)/t + 2(|\alpha| + d)|\rho|/d$ .

**Lemma 6.10.** *For any set  $\{h_\eta\}_{\eta \in \mathcal{X}_j}$  ( $j \geq 0$ ) of complex numbers*

$$(6.16) \quad h_j^*(x) \leq c\mathcal{M}_t\left(\sum_{\eta \in \mathcal{X}_j} |h_\eta| \mathbb{1}_{R_\eta}\right)(x), \quad x \in \mathbb{R}_+^d.$$

Moreover, for  $\xi \in \mathcal{X}_j$

$$(6.17) \quad W_\alpha(A^j; \xi)^{-\rho/d} h_\xi^* \mathbb{1}_{R_\xi}(x) \leq c\mathcal{M}_t\left(\sum_{\eta \in \mathcal{X}_j} |h_\eta| W_\alpha(A^j; \eta)^{-\rho/d} \mathbb{1}_{R_\eta}\right)(x), \quad x \in \mathbb{R}_+^d.$$

Here the constants depend only on  $d, \alpha, \rho, \delta$ , and  $t$ .

**Lemma 6.11.** *Suppose  $g \in V_{4^j}$  and denote*

$$M_\xi := \sup_{x \in R_\xi} |g(x)|, \quad \xi \in \mathcal{X}_j, \quad \text{and} \quad m_\eta := \inf_{x \in R_\eta} |g(x)|, \quad \eta \in \mathcal{X}_{j+\ell}.$$

Then there exists  $\ell \geq 1$ , depending only  $d, \alpha, \delta$ , and  $\lambda$ , such that for any  $\xi \in \mathcal{X}_j$

$$(6.18) \quad M_\xi^* \leq cm_\eta^* \quad \text{for all} \quad \eta \in \mathcal{X}_{j+\ell}, \quad R_\eta \cap R_\xi \neq \emptyset,$$

and, therefore,

$$(6.19) \quad M_\xi^* \mathbb{1}_{R_\xi}(x) \leq c \sum_{\eta \in \mathcal{X}_{j+\ell}, R_\eta \cap R_\xi \neq \emptyset} m_\eta^* \mathbb{1}_{R_\eta}(x), \quad x \in \mathbb{R}^d,$$

where  $c > 0$  depends only on  $d, \alpha, \delta$ , and  $t$ .

**Proof of Theorem 6.7.** Choose  $\sigma$  so that  $\sigma \geq \lambda + 2(|\alpha| + d)|\rho|/d$  and recall that  $t$  has already been selected so that  $0 < t < \min\{p, q\}$ .

Suppose  $\{\Phi_j\}$  are from the definition of Laguerre-Triebel-Lizorkin spaces (see (6.1)–(6.3)). As already mentioned in §5.2, there exists a function  $\widehat{b}$  satisfying (5.18)–(5.19) such that (5.20) holds as well. Using this function we define  $\{\Psi_j\}$  just as in (5.23). Then we use  $\{\Phi_j\}$  and  $\{\Psi_j\}$  to define as in (5.24) a pair of dual needlet systems  $\{\varphi_\eta\}$  and  $\{\psi_\eta\}$ .

Suppose  $\{\tilde{\varphi}_\eta\}, \{\tilde{\psi}_\eta\}$  is a second pair of needlet systems, defined as in (5.22)-(5.24) using another pair of kernels  $\{\tilde{\Phi}_j\}, \{\tilde{\Psi}_j\}$ .

We first show the boundedness of the operator  $T_{\tilde{\psi}} : f_{pq}^{s\rho} \rightarrow F_{pq}^{s\rho}$ . Let  $h \in f_{pq}^{s\rho}$  and set  $f := T_{\tilde{\psi}} h = \sum_{\xi \in \mathcal{X}} h_\xi \tilde{\psi}_\xi$ . Evidently  $\Phi_j * \tilde{\psi}_\xi = 0$  if  $\xi \in \mathcal{X}_m$  and  $|j - m| \geq 2$ , and hence

$$\Phi_j * f = \sum_{m=j-1}^{j+1} \sum_{\xi \in \mathcal{X}_m} h_\xi \Phi_j * \tilde{\psi}_\xi \quad (\mathcal{X}_{-1} := \emptyset).$$

Denote  $H_\xi := h_\xi W_\alpha(4^m; \xi)^{-\rho/d} \mu(R_\xi)^{-1/2}$ . Using Lemma 6.8 and (5.17) we get

$$\begin{aligned} W_\alpha(4^j; x)^{-\rho/d} |\Phi_j * f(x)| &\leq \sum_{m=j-1}^{j+1} \sum_{\xi \in \mathcal{X}_m} |h_\xi| W_\alpha(4^j; x)^{-\rho/d} |\Phi_j * \tilde{\psi}_\xi(x)| \\ (6.20) \quad &\leq c \sum_{m=j-1}^{j+1} \sum_{\xi \in \mathcal{X}_m} \frac{|h_\xi| W_\alpha(4^m; \xi)^{-\rho/d} \mu(R_\xi)^{-1/2}}{(1 + 2^m \|\xi - x\|)^{\sigma - 2(|\alpha| + d)|\rho|/d}} \\ &\leq c \sum_{m=j-1}^{j+1} H_m^*(x) \quad (H_{-1}^* := 0), \end{aligned}$$

where  $H_m^*(x)$  is defined as in (6.14). We use this in the definition of  $\|f\|_{F_{pq}^{s\rho}}$  and apply Lemma 6.10 and the maximal inequality (4.6) to obtain

$$\begin{aligned} \|f\|_{F_{pq}^{s\rho}} &\leq \left\| \left( \sum_{j=0}^{\infty} (2^{js} |H_j^*(\cdot)|)^q \right)^{1/q} \right\|_p \\ &\leq c \left\| \left( \sum_{j=0}^{\infty} \left[ \mathcal{M}_t \left( 2^{js} \sum_{\xi \in \mathcal{X}_j} |h_\xi| W_\alpha(4^j; \xi)^{-\rho/d} \mu(R_\xi)^{-1/2} \mathbb{1}_{R_\xi} \right) \right]^q \right)^{1/q} \right\|_p \\ &\leq c \| \{h_\eta\} \|_{f_{pq}^{s\rho}}. \end{aligned}$$

Hence the operator  $T_{\tilde{\psi}} : f_{pq}^{s\rho} \rightarrow F_{pq}^{s\rho}$  is bounded.

Let the space  $F_{pq}^{s\rho}$  be defined using  $\{\bar{\Phi}_j\}$  instead of  $\{\Phi_j\}$ . We now prove the boundedness of the operator  $S_\varphi : F_{pq}^{s\rho} \rightarrow f_{pq}^{s\rho}$ . Let  $f \in F_{pq}^{s\rho}$  and denote

$$M_\xi := \sup_{x \in R_\xi} |\bar{\Phi}_j * f(x)|, \quad \xi \in \mathcal{X}_j, \quad \text{and} \quad m_\eta := \inf_{x \in R_\eta} |\bar{\Phi}_j * f(x)|, \quad \eta \in \mathcal{X}_{j+\ell},$$

where  $\ell$  is the constant from Lemma 6.11. We have

$$(6.21) \quad |\langle f, \varphi_\xi \rangle| \leq c_\xi^{1/2} |\bar{\Phi}_j * f(\xi)| \leq c \mu(R_\xi)^{1/2} M_\xi \leq c \mu(R_\xi)^{1/2} M_\xi^*.$$

Evidently,  $\bar{\Phi}_j * f \in V_{4^j}$ , and applying Lemma 6.11 (see (6.19)), we get

$$(6.22) \quad M_\xi^* \mathbb{1}_{R_\xi}(x) \leq c \sum_{\eta \in \mathcal{X}_{j+\ell}, R_\eta \cap R_\xi \neq \emptyset} m_\eta^* \mathbb{1}_{R_\eta}(x), \quad x \in \mathbb{R}^d.$$

It is easy to see that  $W_\alpha(4^{j+\ell}; y) \sim W_\alpha(4^j; \xi)$  for  $y \in R_\xi$ . We use this, (6.21)-(6.22), Lemma 6.10, and the maximal inequality (4.6) to obtain

$$\begin{aligned}
\|\langle f, \varphi_\xi \rangle\|_{f_{pq}^{s\rho}} &\leq c \left\| \left( \sum_{j=0}^{\infty} 2^{sjq} \left( \sum_{\xi \in \mathcal{X}_j} W_\alpha(4^j; \xi)^{-\rho/d} M_\xi^* \mathbb{1}_{R_\xi} \right)^q \right)^{1/q} \right\|_p \\
&\leq c \left\| \left( \sum_{j=0}^{\infty} 2^{sjq} \left( \sum_{\eta \in \mathcal{X}_{j+\ell}} W_\alpha(4^{j+\ell}; \eta)^{-\rho/d} m_\eta^* \mathbb{1}_{R_\eta} \right)^q \right)^{1/q} \right\|_p \\
&\leq c \left\| \left( \sum_{j=0}^{\infty} \mathcal{M}_t \left( 2^{sj} \sum_{\eta \in \mathcal{X}_{j+\ell}} W_\alpha(4^{j+\ell}; \eta)^{-\rho/d} m_\eta \mathbb{1}_{R_\eta} \right)^q \right)^{1/q} \right\|_p \\
&\leq c \left\| \left( \sum_{j=0}^{\infty} \left( 2^{sj} \sum_{\eta \in \mathcal{X}_{j+\ell}} W_\alpha(4^{j+\ell}; \eta)^{-\rho/d} m_\eta \mathbb{1}_{R_\eta} \right)^q \right)^{1/q} \right\|_p \\
&\leq c \left\| \left( \sum_{j=0}^{\infty} 2^{sjq} W_\alpha(4^j; \cdot)^{-\rho/d} |\tilde{\Phi}_j * f(\cdot)|^q \right)^{1/q} \right\|_p = c \|f\|_{F_{pq}^{s\rho}}.
\end{aligned}$$

Here for the second inequality we used that each tile  $R_\eta$ ,  $\eta \in \mathcal{X}_{j+l}$ , intersects no more than finitely many (depending only on  $d$ ) tiles  $R_\eta$ ,  $\eta \in \mathcal{X}_j$ . The above estimates prove the boundedness of the operator  $S_\varphi : F_{pq}^{s\rho} \rightarrow f_{pq}^{s\rho}$ . The identity  $T_\psi \circ S_\varphi = Id$  follows by Theorem 5.3.

It remains to show the independence of the definition of Triebel-Lizorkin spaces from the specific selection of  $\hat{a}$  satisfying (6.2)-(6.3). Suppose  $\{\Phi_j\}$ ,  $\{\tilde{\Phi}_j\}$  are two sequences of kernels as in the definition of Triebel-Lizorkin spaces defined by two different functions  $\hat{a}$  satisfying (6.2)-(6.3). As above there exist two associated needlet systems  $\{\Phi_j\}$ ,  $\{\Psi_j\}$ ,  $\{\varphi_\xi\}$ ,  $\{\psi_\xi\}$  and  $\{\tilde{\Phi}_j\}$ ,  $\{\tilde{\Psi}_j\}$ ,  $\{\tilde{\varphi}_\xi\}$ ,  $\{\tilde{\psi}_\xi\}$ . Denote by  $\|f\|_{F_{pq}^{s\rho}(\Phi)}$  and  $\|f\|_{F_{pq}^{s\rho}(\tilde{\Phi})}$  the  $F$ -norms defined via  $\{\Phi_j\}$  and  $\{\tilde{\Phi}_j\}$ . Then from above

$$\|f\|_{F_{pq}^{s\rho}(\Phi)} \leq c \|\langle f, \tilde{\varphi}_\xi \rangle\|_{f_{pq}^{s\rho}} \leq c \|f\|_{F_{pq}^{s\rho}(\tilde{\Phi})}.$$

The independence of the definition of  $F_{pq}^{s\rho}$  of the specific choice of  $\hat{a}$  in the definition of the functions  $\{\Phi_j\}$  follows by interchanging the roles of  $\{\Phi_j\}$  and  $\{\tilde{\Phi}_j\}$  and their complex conjugates.  $\square$

To us the spaces  $F_{pq}^{ss}$  are more natural than the spaces  $F_{pq}^{s\rho}$  with  $\rho \neq s$  since they embed correctly with respect to the smoothness index  $s$ .

**Proposition 6.12.** *Let  $0 < p < p_1 < \infty$ ,  $0 < q, q_1 \leq \infty$ , and  $-\infty < s_1 < s < \infty$ . Then we have the continuous embedding*

$$(6.23) \quad F_{pq}^{ss} \subset F_{p_1 q_1}^{s_1 s_1} \quad \text{if} \quad s/d - 1/p = s_1/d - 1/p_1.$$

The proof of this embedding result can be carried out similarly as the proof of Proposition 4.11 in [10], using the idea of the proof in the classical case on  $\mathbb{R}^n$  (see e.g. [18], page 129). We omit it.

## 7. LAGUERRE-BESOV SPACES

We introduce weighted Besov spaces on  $\mathbb{R}_+^d$  in the context of Laguerre expansions using the kernels  $\{\Phi_j\}$  from (6.1) with  $\hat{a}$  satisfying (6.2)-(6.3) (see [13], [18] for the general idea of using orthogonal or spectral decompositions in defining Besov spaces).

### 7.1. Definition of Laguerre-Besov Spaces.

**Definition 7.1.** Let  $s, \rho \in \mathbb{R}$  and  $0 < p, q \leq \infty$ . The Laguerre-Besov space  $B_{pq}^{s\rho} := B_{pq}^{s\rho}(\mathcal{F}^\alpha)$  is defined as the set of all  $f \in \mathcal{S}'_+$  such that

$$(7.1) \quad \|f\|_{B_{pq}^{s\rho}} := \left( \sum_{j=0}^{\infty} \left( 2^{sj} \|W_\alpha(4^j; \cdot)^{-\rho/d} \Phi_j * f(\cdot)\|_p \right)^q \right)^{1/q} < \infty,$$

where the  $\ell_q$ -norm is replaced by the sup-norm if  $q = \infty$ .

Observe that as in the case of Laguerre-Triebel-Lizorkin spaces the above definition is independent of the particular choice of  $\hat{a}$  obeying (6.2)-(6.3) (see Theorem 7.4). Also, as for  $F_{pq}^{s\rho}$  the Besov space  $B_{pq}^{s\rho}$  is a quasi-Banach space which is continuously embedded in  $\mathcal{S}'_+$ . We skip the details.

**7.2. Needlet Decomposition of Laguerre-Besov Spaces.** We next define the sequence spaces  $b_{pq}^{s\rho}$  associated to the Laguerre-Besov spaces  $B_{pq}^{s\rho}$ . As in §6 we assume that  $\{\mathcal{X}_j\}_{j=0}^{\infty}$  are from (5.9) with associated tiles  $\{R_\xi\}_{\xi \in \mathcal{X}_j}$  from (5.11). As before we set  $\mathcal{X} := \cup_{j \geq 0} \mathcal{X}_j$ .

**Definition 7.2.** Let  $s, \rho \in \mathbb{R}$  and  $0 < p, q \leq \infty$ . Then  $b_{pq}^{s\rho}$  is defined to be the space of all complex-valued sequences  $h := \{h_\xi\}_{\xi \in \mathcal{X}}$  such that

$$(7.2) \quad \|h\|_{b_{pq}^{s\rho}} := \left( \sum_{j=0}^{\infty} 2^{jsq} \left[ \sum_{\xi \in \mathcal{X}_j} \left( W_\alpha(4^j; \xi)^{-\rho/d} \mu(R_\xi)^{1/p-1/2} |h_\xi| \right)^p \right]^{q/p} \right)^{1/q}$$

is finite, with the usual modification whenever  $p = \infty$  or  $q = \infty$ .

We shall utilize again the analysis and synthesis operators  $S_\varphi$  and  $T_\psi$  defined in (6.8). The next lemma guarantees that the operator  $T_\psi$  is well defined on  $b_{pq}^{s\rho}$ .

**Lemma 7.3.** Let  $s, \rho \in \mathbb{R}$  and  $0 < p, q \leq \infty$ . Then for any  $h \in b_{pq}^{s\rho}$ ,  $T_\psi h := \sum_{\xi \in \mathcal{X}} h_\xi \psi_\xi$  converges in  $\mathcal{S}'_+$ . Moreover, the operator  $T_\psi : b_{pq}^{s\rho} \rightarrow \mathcal{S}'_+$  is continuous.

The proof of this lemma is quite similar to the proof of Lemma 6.6 and will be omitted.

Our main result in this section is the following characterization of Laguerre-Besov spaces.

**Theorem 7.4.** Let  $s, \rho \in \mathbb{R}$  and  $0 < p, q \leq \infty$ . Then the operators  $S_\varphi : B_{pq}^{s\rho} \rightarrow b_{pq}^{s\rho}$  and  $T_\psi : b_{pq}^{s\rho} \rightarrow B_{pq}^{s\rho}$  are bounded and  $T_\psi \circ S_\varphi = Id$  on  $B_{pq}^{s\rho}$ . Consequently, for  $f \in \mathcal{S}'_+$  we have that  $f \in B_{pq}^{s\rho}$  if and only if  $\{\langle f, \varphi_\xi \rangle\}_{\xi \in \mathcal{X}} \in b_{pq}^{s\rho}$  and

$$(7.3) \quad \|f\|_{B_{pq}^{s\rho}} \sim \|\{\langle f, \varphi_\xi \rangle\}_{\xi \in \mathcal{X}}\|_{b_{pq}^{s\rho}}.$$

In addition, the definition of  $B_{pq}^{s\rho}$  is independent of the particular selection of  $\hat{a}$  satisfying (6.2)-(6.3).

The proof of this theorem relies on some lemmas from the proof of Theorem 6.7 as well as the next lemma with proof given in Section 9.

**Lemma 7.5.** Let  $0 < p \leq \infty$  and  $\rho \in \mathbb{R}$ . Then for any  $g \in V_{4^j}$ ,  $j \geq 0$ ,

$$(7.4) \quad \left( \sum_{\xi \in \mathcal{X}_j} W_\alpha(4^j; \xi)^{-\rho p/d} \max_{x \in R_\xi} |g(x)|^p \mu(R_\xi) \right)^{1/p} \leq c \|W_\alpha(4^j; \cdot)^{-\rho/d} g(\cdot)\|_p.$$

**Proof of Theorem 7.4.** We will use some basic assumptions and notation from the proof of Theorem 6.7. Let  $0 < t < p$  and  $\sigma \geq \lambda + 2(|\alpha| + d)|\rho|/d$ . Assume that  $\{\Phi_j\}, \{\Psi_j\}, \{\varphi_\eta\}, \{\psi_\eta\}$  and  $\{\tilde{\Phi}_j\}, \{\tilde{\Psi}_j\}, \{\tilde{\varphi}_\eta\}, \{\tilde{\psi}_\eta\}$  are two needlet systems, defined as in (5.22)-(5.24), that originate from two completely different functions  $\hat{a}$  satisfying (6.2)-(6.3).

Let us first prove the boundedness of the operator  $T_{\tilde{\psi}} : b_{pq}^{s\rho} \rightarrow B_{pq}^{s\rho}$ , assuming that  $B_{pq}^{s\rho}$  is defined by  $\{\tilde{\Phi}_j\}$ . Suppose  $h \in b_{pq}^{s\rho}$  and set  $f := T_{\tilde{\psi}}h = \sum_{\xi \in \mathcal{X}} h_\xi \tilde{\psi}_\xi$ .

Denote  $H_\xi := h_\xi W_\alpha(4^m; \xi)^{-\rho/d} \mu(R_\xi)^{-1/2}$ ,  $\xi \in \mathcal{X}_m$ . Then by (6.20) and Lemma 6.10

$$\begin{aligned} \|W_\alpha(4^j; \cdot)^{-\rho/d} \tilde{\Phi}_j * f(\cdot)\|_p &\leq c \sum_{m=j-1}^{j+1} \|H_m^*\|_p \\ &\leq c \sum_{m=j-1}^{j+1} \left\| \mathcal{M}_t \left( \sum_{\xi \in \mathcal{X}_m} |h_\xi| W_\alpha(4^m; \xi)^{-\rho/d} \mu(R_\xi)^{-1/2} \mathbb{1}_{R_\xi} \right) \right\|_p \\ &\leq c \sum_{m=j-1}^{j+1} \left( \sum_{\xi \in \mathcal{X}_m} \left( |h_\xi| W_\alpha(4^m; \xi)^{-\rho/d} \mu(R_\xi)^{1/p-1/2} \right)^p \right)^{1/p}, \end{aligned}$$

which yields  $\|f\|_{B_{pq}^{s\rho}} \leq c \|\{h_\eta\}\|_{b_{pq}^{s\rho}}$  and hence the claimed boundedness of  $T_{\tilde{\psi}}$ .

We now prove the boundedness of the operator  $S_\varphi : B_{pq}^{s\rho} \rightarrow b_{pq}^{s\rho}$ , where we assume that the space  $B_{pq}^{s\rho}$  is defined in terms of  $\{\tilde{\Phi}_j\}$  in place of  $\{\Phi_j\}$ . Just as in (6.21) we have  $|\langle f, \varphi_\xi \rangle| \leq c \mu(R_\xi)^{1/2} |\tilde{\Phi}_j * f(\xi)|$ ,  $\xi \in \mathcal{X}_j$ . Since  $\tilde{\Phi}_j * f \in V_{4^j}$ , Lemma 7.5 implies

$$\begin{aligned} &\sum_{\xi \in \mathcal{X}_j} \left( W_\alpha(4^j; \xi)^{-\rho/d} \mu(R_\xi)^{1/p-1/2} |\langle f, \varphi_\xi \rangle| \right)^p \\ &\leq c \sum_{\xi \in \mathcal{X}_j} W_\alpha(4^j; \xi)^{-\rho p/d} |\tilde{\Phi}_j * f(\xi)|^p \mu(R_\xi) \leq c \|W_\alpha(4^j; \cdot)^{-\rho/d} \tilde{\Phi}_j * f(\cdot)\|_p^p, \end{aligned}$$

which leads immediately to  $\|\langle f, \varphi \rangle\|_{b_{pq}^{s\rho}} \leq c \|f\|_{B_{pq}^{s\rho}}$ .

The identity  $T_{\tilde{\psi}} \circ S_\varphi = Id$  follows by Proposition 5.3. The independence of  $B_{pq}^{s\rho}$  of the specific selection of  $\hat{a}$  in the definition of  $\{\tilde{\Phi}_j\}$  follows from above exactly as in the Triebel-Lizorkin case (see the proof of Theorem 6.7).  $\square$

The parameter  $\rho$  in the definition of the Besov spaces  $B_{pq}^{s\rho}$  allows one to consider various scales of spaces. A ‘‘classical’’ choice of  $\rho$  would be  $\rho = 0$ . However, to us most natural are the spaces  $B_{pq}^{ss}$  ( $\rho = s$ ) for they embed ‘‘correctly’’ with respect to the smoothness index  $s$ :

**Proposition 7.6.** *Let  $0 < p \leq p_1 \leq \infty$ ,  $0 < q \leq q_1 \leq \infty$ , and  $-\infty < s_1 \leq s < \infty$ . Then we have the continuous embedding*

$$(7.5) \quad B_{pq}^{ss} \subset B_{p_1 q_1}^{s_1 s_1} \quad \text{if} \quad s/d - 1/p = s_1/d - 1/p_1.$$

**Proof.** Assuming that  $\Phi_j$  is from Definition 7.1 we have  $\Phi_j * f \in V_{4^{j+1}}$  and applying estimate (4.2) from Proposition 4.1 we obtain

$$\|W_\alpha(4^j; \cdot)^{-s_1/d} \Phi_j * f(\cdot)\|_{p_1} \leq c 2^{jd(1/p-1/p_1)} \|W_\alpha(4^j; \cdot)^{-s/d} \Phi_j * f(\cdot)\|_p,$$

where we used that  $s/d - 1/p = s_1/d - 1/p_1$ . This implies (7.5) at once.  $\square$

## 8. PROOFS FOR SECTIONS 3-4

8.1. **Proof of estimates (3.10) and (3.12) in Theorem 3.2.** We may assume that  $n \geq n_0$ , where  $n_0$  is a sufficiently large constant. Estimate (3.10) will be established by applying repeatedly summation by parts to the sum in the definition (3.7) of  $\Lambda_n(x, y)$ . For a sequence of numbers  $\{a_m\}$  we denote by  $\Delta^k a_m$  the  $k$ th forward differences, defined by  $\Delta a_m := a_m - a_{m+1}$  and inductively  $\Delta^{k+1} a_m := \Delta(\Delta^k a_m)$ . Choose  $k \geq \sigma + 2|\alpha| + d/2$  and denote

$$(8.1) \quad \Omega_n^k(x, y) := \sum_{m=0}^n A_{n-m}^k \mathcal{F}_m^\alpha(x, y), \quad A_m^k := \binom{m+k}{k}.$$

Using summation by parts  $k$  times, we obtain

$$(8.2) \quad \Lambda_n(x, y) := \sum_{j=0}^{\infty} \widehat{a}\left(\frac{j}{n}\right) \mathcal{F}_j^\alpha(x, y) = \sum_{m=0}^{\infty} \Delta^{k+1} \widehat{a}\left(\frac{m}{n}\right) \cdot \Omega_m^k(x, y),$$

where  $\Delta^{k+1}$  is applied with respect to  $m$ . By (2.1) and (8.1), it easily follows that  $\Omega_m^k(x, y) = c e^{-(\|x\|_2^2 + \|y\|_2^2)/2} P_m^{\alpha, k}(x^2, y^2)$  and combining this with (2.4) we get

$$\begin{aligned} \Omega_m^k(x, y) &= c \int_{[0, \pi]^d} L_m^{|\alpha|+k+d} \left( \|x\|_2^2 + \|y\|_2^2 + 2 \sum_{i=1}^d x_i y_i \cos \theta_i \right) \\ &\quad \times e^{-(\|x\|_2^2 + \|y\|_2^2 + 2 \sum_{i=1}^d x_i y_i \cos \theta_i)/2} \prod_{i=1}^d j_{\alpha_i - 1/2}(x_i y_i \cos \theta_i) \sin^{2\alpha_i} \theta_i d\theta. \end{aligned}$$

Using this in (8.2) we arrive at the identity

$$(8.3) \quad \begin{aligned} \Lambda_n(x, y) &= c \int_{[0, \pi]^d} \mathcal{K}_n^\lambda \left( \|x\|^2 + \|y\|^2 + 2 \sum_{i=1}^d x_i y_i \cos \theta_i \right) \\ &\quad \times \prod_{i=1}^d j_{\alpha_i - 1/2}(x_i y_i \cos \theta_i) \sin^{2\alpha_i} \theta_i d\theta, \end{aligned}$$

where  $\lambda := |\alpha| + k + d$  and the kernel  $\mathcal{K}_n^\lambda$  is defined by

$$(8.4) \quad \mathcal{K}_n^\lambda(t) := \sum_{m=0}^{\infty} \Delta^{k+1} \widehat{a}\left(\frac{m}{n}\right) L_m^\lambda(t) e^{-t/2}.$$

By a well known property of finite differences we have

$$(8.5) \quad \left| \Delta^{k+1} \widehat{a}\left(\frac{m}{n}\right) \right| = n^{-k-1} |\widehat{a}^{(k+1)}(\xi)| \leq n^{-k-1} \|\widehat{a}^{(k+1)}\|_{L^\infty}.$$

Further, it is known that [1, p. 204]

$$(8.6) \quad j_{\alpha - \frac{1}{2}}(x) = x^{-\alpha + 1/2} J_{\alpha - 1/2}(x) = \frac{2^{-\alpha + \frac{1}{2}}}{\sqrt{\pi} \Gamma(\alpha)} \int_{-1}^1 e^{ixt} (1-t^2)^{\alpha-1} dt, \quad \alpha > 0,$$

and  $j_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \cos x$ . Therefore,

$$(8.7) \quad |j_{\alpha - \frac{1}{2}}(x)| \leq c_\alpha < \infty, \quad x \in \mathbb{R}_+, \alpha \geq 0.$$

By (8.5) and (2.11) (with  $\alpha$  replaced by  $|\alpha| + k + 1$ ) we obtain for  $t > 0$

$$(8.8) \quad |\mathcal{K}_n^\lambda(t)| \leq c \sum_{m=\max\{\lfloor un \rfloor - k, 1\}}^{\lfloor (1+v)n \rfloor} \frac{1}{m^{k+1}} \left(\frac{m}{t}\right)^{(|\alpha|+k+d)/2} \leq cn^{(-k+|\alpha|+d)/2} t^{-(|\alpha|+k+d)/2}.$$

Using this in (8.3) we get

$$|\Lambda_n(x, y)| \leq cn^{(-k+|\alpha|+d)/2} \int_{[0, \pi]^d} \frac{\prod_{i=1}^d \sin^{2\alpha_i} \theta_i d\theta}{\left(\|x\|_2^2 + \|y\|_2^2 + 2 \sum_{i=1}^d x_i y_i \cos \theta_i\right)^{(k+|\alpha|+d)/2}}.$$

Set  $\tau := (k + |\alpha| + d)/2$ . Substituting  $\theta_i = \pi - t_i$  in the above integral and using  $1 - \cos t = 2 \sin^2 \frac{t}{2} \sim t^2$  we infer

$$(8.9) \quad |\Lambda_n(x, y)| \leq cn^{-k+\tau} \int_{[0, \pi]^d} \frac{\prod_{i=1}^d \sin^{2\alpha_i} t_i dt}{\left(\|x - y\|_2^2 + 4 \sum_{i=1}^d x_i y_i \sin^2 \frac{t_i}{2}\right)^\tau} \\ \leq cn^{-k+\tau} \int_{[0, \pi]^d} \frac{\prod_{i=1}^d t_i^{2\alpha_i} dt}{\left(\|x - y\|^2 + \sum_{i=1}^d x_i y_i t_i^2\right)^\tau} =: cM_n^{k, \alpha}(x, y).$$

We estimate the integral above in two ways. First, we trivially have

$$(8.10) \quad |\Lambda_n(x, y)| \leq cM_n^{k, \alpha}(x, y) \leq \frac{cn^{-k+\tau}}{\|x - y\|^{2\tau}} \leq \frac{cn^{|\alpha|+d}}{(n^{1/2}\|x - y\|)^{k+|\alpha|+d}}.$$

The second estimate is really many estimates rolled into one. For a fixed  $1 \leq \ell \leq d$  we partition  $\alpha$  into  $\alpha = (\alpha', \alpha'')$  with  $\alpha' = (\alpha_1, \dots, \alpha_\ell)$  and  $\alpha'' = (\alpha_{\ell+1}, \dots, \alpha_d)$ . Since  $\tau > |\alpha| + d/2$  and  $x_i y_i > 0$  we have

$$M_n^{k, \alpha}(x, y) \leq cn^{-k+\tau} \int_{[0, \pi]^\ell} \frac{\prod_{i=1}^\ell t_i^{2\alpha_i} dt}{\left(\|x - y\|^2 + \sum_{i=1}^\ell x_i y_i t_i^2\right)^\tau} \\ \leq \frac{cn^{-k+\tau}}{\prod_{i=1}^\ell (x_i y_i)^{\alpha_i + 1/2}} \prod_{i=1}^\ell \int_0^{\pi(x_i y_i)^{1/2}} \frac{du}{\left(\|x - y\|^2 + \sum_{i=1}^\ell u_i^2\right)^{\tau - |\alpha'|}},$$

where we applied the substitutions  $u_i = t_i(x_i y_i)^{1/2}$  and used  $|\alpha'|$  power of the main term in the denominator to cancel the numerator. Enlarging the integral domain to  $\mathbb{R}^\ell$  and using polar coordinates, the above product of integrals is bounded by

$$\int_{\mathbb{R}^\ell} \frac{du}{\left(\|x - y\|^2 + \|u\|_2^2\right)^{\tau - |\alpha'|}} = \int_0^\infty \frac{r^{\ell-1} dr}{\left(\|x - y\|^2 + r^2\right)^{\tau - |\alpha'|}} \leq \frac{c}{\|x - y\|^{2(\tau - |\alpha'|) - \ell}}.$$

From above and a little algebra we obtain for  $1 \leq \ell \leq d$

$$(8.11) \quad |\Lambda_n(x, y)| \leq cM_n^{k, \alpha}(x, y) \\ \leq \frac{cn^{d/2}}{\prod_{i=1}^\ell (x_i y_i)^{\alpha_i + \frac{1}{2}} \prod_{i=\ell+1}^d (n^{-1})^{\alpha_i + \frac{1}{2}} (n^{1/2}\|x - y\|)^{k+|\alpha| - 2|\alpha'| + d - \ell}}.$$

A third bound on  $|\Lambda_n(x, y)|$  will be obtained by estimating all terms in (3.7). By (2.10) and (3.6) it follows that

$$(8.12) \quad \|\mathcal{F}_{\nu_i}^{\alpha_i}\|_\infty \leq c\nu_i^{\alpha_i/2}, \quad 1 \leq i \leq d, \quad \text{and} \quad \|\mathcal{F}_\nu^\alpha\|_\infty \leq c\nu^{\alpha/2},$$



and hence

$$|\mathcal{F}_m^\alpha(x, y)| \leq c \sum_{|\nu|=m} \nu^\alpha = c \binom{m+d-1}{m} m^{|\alpha|} \leq cm^{|\alpha|+d-1}, \quad \text{yielding}$$

$$(8.13) \quad |\Lambda_n(x, y)| \leq c \sum_{m=0}^{\lfloor (1+v)n \rfloor} m^{|\alpha|+d-1} \leq cn^{|\alpha|+d}.$$

We also need the estimate:

$$(8.14) \quad |\Lambda_n(x, y)| \leq \frac{cn^{d/2}}{\prod_{i=1}^{\ell} (x_i y_i)^{\alpha_i+1/2} \prod_{i=\ell+1}^d (n^{-1})^{\alpha_i+1/2}}, \quad 1 \leq \ell \leq d.$$

By (2.8) it follows that

$$(8.15) \quad |\mathcal{F}_n^\alpha(x)| \leq \frac{c}{x^{\alpha+1/2} n^{1/4}}, \quad \text{if } x^2 \in \mathbb{R}_+ \setminus (2n+2\alpha+2, 6n+3\alpha+3),$$

and if  $x^2 \in [2n+2\alpha+2, 6n+3\alpha+3]$  by (2.9)

$$(8.16) \quad |\mathcal{F}_n^\alpha(x)| \leq \frac{c}{x^\alpha n^{1/4} (n^{1/3} + |4n+2\alpha+2-x^2|)^{1/4}}.$$

From these two estimates one easily concludes that for  $x > 0$

$$(8.17) \quad |\mathcal{F}_n^\alpha(x)| \leq \frac{c}{x^{\alpha+1/2} n^{1/4}}, \quad \text{if } n \in \mathbb{R}_+ \setminus (x^2/5, x^2/3),$$

and

$$(8.18) \quad |\mathcal{F}_n^\alpha(x)| \leq \frac{c}{x^{\alpha+1/2} (1 + |4n-x^2|)^{1/4}}, \quad \text{if } n \in [x^2/5, x^2/3].$$

Hence,  $|\mathcal{F}_n^\alpha(x)|$  can be bounded by the sum of the right-hand-side quantities in (8.17)-(8.18). Also, from (8.12)  $\|\mathcal{F}_{\nu_i}^{\alpha_i}\|_\infty \leq c\nu_i^{\alpha_i/2}$ . From these along with (3.3)-(3.4) we obtain

$$\begin{aligned} |\Lambda_n(x, y)| &\leq \prod_{i=1}^d \sum_{\nu_i=0}^{\lfloor (1+v)n \rfloor} |\mathcal{F}_{\nu_i}^{\alpha_i}(x_i)| |\mathcal{F}_{\nu_i}^{\alpha_i}(y_i)| \leq \frac{c}{\prod_{i=1}^{\ell} (x_i y_i)^{\alpha_i+1/2}} \prod_{i=\ell+1}^d \sum_{\nu_i=0}^{\lfloor (1+v)n \rfloor} (\nu_i+1)^{\alpha_i} \\ &\times \prod_{i=1}^{\ell} \sum_{\nu_i=0}^{\lfloor (1+v)n \rfloor} \left( \frac{1}{(1+\nu_i)^{1/4}} + \frac{1}{(1+|\nu_i-u_i|)^{1/4}} \right) \left( \frac{1}{(1+\nu_i)^{1/4}} + \frac{1}{(1+|\nu_i-v_i|)^{1/4}} \right), \end{aligned}$$

where  $u_i, v_i > 0$  are some numbers. Clearly, each of the last sums can be bounded by four sums of the form

$$\sum_{\nu_i=0}^{\lfloor (1+v)n \rfloor} \frac{1}{(1+|\nu_i-w_i|)^{1/4} (1+|\nu_i-z_i|)^{1/4}} \leq cn^{1/2}.$$

This last estimate apparently holds independently of  $w_i$  and  $z_i$ . Estimate (8.14) follows from above.

We are now in a position to complete the proof of (3.10). Estimates (8.10) and (8.13) readily imply

$$(8.19) \quad |\Lambda_n(x, y)| \leq \frac{cn^{|\alpha|+d}}{(1+n^{1/2}\|x-y\|)^{k+|\alpha|+d}},$$

while by (8.11) and (8.14) we have for  $1 \leq i \leq \ell$

$$|\Lambda_n(x, y)| \leq \frac{cn^{d/2}}{\prod_{i=1}^{\ell} (x_i y_i)^{\alpha_i+1/2} \prod_{i=\ell+1}^d (n^{-1})^{\alpha_i+1/2} (1+n^{1/2}\|x-y\|)^{k-|\alpha|}}.$$

Clearly, this estimate holds for an arbitrary permutation  $i_1, i_2, \dots, i_d$  of the indices  $1, 2, \dots, d$ . These estimates and (8.19) yield

$$(8.20) \quad |\Lambda_n(x, y)| \leq \frac{cn^{d/2}}{\prod_{i=1}^d (x_i y_i + n^{-1})^{\alpha_i+1/2} (1+n^{1/2}\|x-y\|)^{k-|\alpha|}}.$$

To complete the proof we need the following simple inequality: For  $x, y \in \mathbb{R}_+^d$

$$(8.21) \quad (x_i + n^{-1/2})(y_i + n^{-1/2}) \leq 3(x_i y_i + n^{-1})(1 + n^{1/2}\|x - y\|), \quad 1 \leq i \leq d.$$

Combining these with (8.20) we get

$$|\Lambda_n(x, y)| \leq \frac{cn^{d/2}}{\prod_{i=1}^d (x_i + n^{-1/2})^{\alpha_i+1/2} (y_i + n^{-1/2})^{\alpha_i+1/2} (1+n^{1/2}\|x-y\|)^{k-2|\alpha|-d/2}},$$

which implies (3.10) since  $k$  was select so that  $k \geq \sigma + 2|\alpha| + d/2$ .

The proof of (3.12) is trivial. Indeed, by Lemma 2.1 it follows that

$$(8.22) \quad |\mathcal{F}_n^\alpha(x)| \leq cx^{-\alpha} e^{-\gamma x^2} \quad \text{for } x \geq (6(1+v)n + 3\alpha + 3)^{1/2}.$$

From this it easily follows that if  $\max\{\|x\|^2, \|y\|^2\} \geq (6(1+v)n + 3\alpha + 3)^{1/2}$ , then

$$|\Lambda_n(x, y)| \leq cn^d e^{-\gamma \max\{\|x\|^2, \|y\|^2\}}, \quad \gamma > 0,$$

which readily implies (3.12).  $\square$

**8.2. Proof of estimates (3.11) and (3.13) in Theorem 3.2.** Clearly, (3.13) implies (3.11) if  $\max\{\|x\|, \|y\|\} \geq (6(1+v)n + 3\alpha + 3)^{1/2}$ .

Assume  $\max\{\|x\|, \|y\|\} < (6(1+v)n + 3\alpha + 3)^{1/2} \leq cn^{1/2}$ . We will prove (3.11) in this case by using the scheme of the proof of (3.10) with appropriate modifications. First, we need information about the derivative of  $\mathcal{F}_n^\alpha$ . The Laguerre polynomials satisfy the relation [16, (5.1.14)]

$$(8.23) \quad \frac{d}{dx} L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x) = x^{-1} [nL_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x)].$$

After taking the derivative of  $\mathcal{F}_n^\alpha$  (see (3.1)), the first identity in (8.23) yields

$$(8.24) \quad \frac{d}{dx} \mathcal{F}_n^\alpha(x) = -x [\mathcal{F}_n^\alpha(x) + 2\sqrt{n}\mathcal{F}_{n-1}^{\alpha+1}(x)],$$

and from the second identity we similarly get

$$(8.25) \quad x \frac{d}{dx} \mathcal{F}_n^\alpha(x) = -x^2 \mathcal{F}_n^\alpha(x) + 2n\mathcal{F}_n^\alpha(x) - 2b_n \mathcal{F}_{n-1}^\alpha, \quad b_n := \sqrt{n(n+\alpha)}.$$

Here and in what follows we assume  $\mathcal{F}_k^\alpha(x) = 0$  for  $k < 0$ . Also, from the recurrence relation for Laguerre polynomials [16, (5.1.10)] one readily derives the identity

$$xL_n^\alpha(x) = (2n + \alpha + 1)L_n^\alpha(x) - (n + 1)L_{n+1}^\alpha - (n + \alpha)L_{n-1}^\alpha(x), \quad n \geq 1,$$

with  $L_0^\alpha(x) = 1$  and  $L_1^\alpha(x) = -x + \alpha + 1$ . From this with the definition of  $\mathcal{F}_n^\alpha$  in (3.1), we get

$$(8.26) \quad x^2 \mathcal{F}_n^\alpha(x) = -b_{n+1} \mathcal{F}_{n+1}^\alpha(x) + (2n + \alpha + 1) \mathcal{F}_n^\alpha(x) - b_n \mathcal{F}_{n-1}^\alpha(x),$$

where  $b_n$  is as above. Combining this with (8.25) gives

$$(8.27) \quad \frac{d}{dx} \mathcal{F}_n^\alpha(x) = x^{-1} [ -(\alpha + 1) \mathcal{F}_n^\alpha(x) + b_{n+1} \mathcal{F}_{n+1}^\alpha(x) - b_n \mathcal{F}_{n-1}^\alpha(x) ].$$

We also need the relation [16, (5.1.13)]

$$(8.28) \quad L_n^\alpha(x) = L_n^{\alpha+1}(x) - L_{n-1}^{\alpha+1}(x).$$

From this and (3.1) we deduce

$$(8.29) \quad \mathcal{F}_n^\alpha(x) = \sqrt{n + \alpha + 1} \mathcal{F}_n^{\alpha+1}(x) - \sqrt{n} \mathcal{F}_{n-1}^{\alpha+1}(x).$$

Using this identity with  $\alpha$  replaced by  $\alpha - 1$ , (8.27), and the obvious fact that  $b_n = n + \mathcal{O}(1)$ , we arrive at

$$(8.30) \quad \left| \frac{d}{dx} \mathcal{F}_n^\alpha(x) \right| \leq cx^{-1} \left[ \max_{n-1 \leq m \leq n+1} |\mathcal{F}_m^\alpha(x)| + n^{1/2} \max_{n \leq m \leq n+1} |\mathcal{F}_m^{\alpha-1}(x)| \right].$$

By (8.24) and (8.12) we readily get the estimate  $\left| \frac{d}{dx} \mathcal{F}_n^\alpha(x) \right| \leq c n^{\alpha/2+1}$  and by (8.30) and (8.12)  $\left| \frac{d}{dx} \mathcal{F}_n^\alpha(x) \right| \leq cx^{-1} n^{\alpha/2}$ . Therefore,

$$(8.31) \quad \left| \frac{d}{dx} \mathcal{F}_n^\alpha(x) \right| \leq cn^{\alpha/2} \min\{x^{-1}, nx\} \leq cn^{(\alpha+1)/2}, \quad x \in \mathbb{R}_+.$$

We use this estimate to obtain

$$(8.32) \quad \begin{aligned} \left| \frac{\partial}{\partial x_r} \Lambda_n(x, y) \right| &\leq \sum_{m=0}^{\lfloor (1+\nu)n \rfloor} \sum_{|\nu|=m} |\mathcal{F}_\nu^\alpha(y)| \left| \frac{\partial}{\partial x_r} \mathcal{F}_\nu^\alpha(x) \right| \\ &\leq cn^{1/2} \sum_{m=0}^{\lfloor (1+\nu)n \rfloor} \sum_{|\nu|=m} \nu^\alpha \leq cn^{|\alpha|+d+1/2}. \end{aligned}$$

We next prove an analogue of (8.14). Let  $0 < x \leq cn^{1/2}$ . Assuming that  $m \in \mathbb{R} \setminus (x^2/5, x^2/3)$  we derive as before from (8.15) and (8.24)

$$(8.33) \quad \begin{aligned} \left| \frac{d}{dx} \mathcal{F}_m^\alpha(x) \right| &\leq x(|\mathcal{F}_m^\alpha(x)| + 2m^{1/2} |\mathcal{F}_{m-1}^{\alpha+1}(x)|) \\ &\leq cx \left( \frac{1}{x^{\alpha+1/2} m^{1/4}} + \frac{m^{1/2}}{x^{\alpha+3/2} m^{1/4}} \right) \leq \frac{cn^{1/2}}{x^{\alpha+1/2} m^{1/4}}. \end{aligned}$$

From (8.16) and (8.24) we similarly obtain

$$(8.34) \quad \left| \frac{d}{dx} \mathcal{F}_m^\alpha(x) \right| \leq \frac{cn^{1/2}}{x^{\alpha+1/2} (1 + |4m - x^2|)^{1/4}} \quad \text{for } m \in (x^2/5, x^2/3).$$

We further proceed exactly as in the proof of (8.14), with  $\mathcal{F}_{\nu_r}^{\alpha_r}(x_r)$  replaced by  $\frac{\partial}{\partial x_r} \mathcal{F}_{\nu_r}^{\alpha_r}(x_r)$  and for this term estimates (8.17)-(8.18) are replaced by (8.33)-(8.34), and we also use (8.31). As a result, we get

$$(8.35) \quad \left| \frac{\partial}{\partial x_r} \Lambda_n(x, y) \right| \leq \frac{cn^{(d+1)/2}}{\prod_{i=1}^\ell (x_i y_i)^{\alpha_i+1/2} \prod_{i=\ell+1}^d (n^{-1})^{\alpha_i+1/2}}, \quad 1 \leq \ell \leq d.$$

We now derive our main bound on  $|(\partial/\partial x_r) \Lambda_n(x, y)|$ . It will be convenient to use the notation  $\partial f(t) := f'(t)$ . After differentiating the expression of  $\Lambda_n(x, y)$  in

(8.3) we obtain for  $1 \leq r \leq d$ ,

$$(8.36) \quad \frac{\partial}{\partial t_r} \Lambda_n(x, y) = \mathcal{Q}_1(x, y) + \mathcal{Q}_2(x, y),$$

where

$$(8.37) \quad \mathcal{Q}_1(x, y) := \int_{[0, \pi]^d} \partial \mathcal{K}_n^{k+|\alpha|+d} \left( \|x\|_2^2 + \|y\|_2^2 + 2 \sum_{i=1}^d x_i y_i \cos \theta_i \right) \\ \times (2x_r - 2y_r \cos \theta_r) \prod_{i=1}^d j_{\alpha_i - \frac{1}{2}}(x_i y_i \cos \theta_i) (\sin \theta_i)^{2\alpha_i} d\theta,$$

$$(8.38) \quad \mathcal{Q}_2(x, y) := \int_{[0, \pi]^d} \mathcal{K}_n^{k+|\alpha|+d} \left( \|x\|_2^2 + \|y\|_2^2 + 2 \sum_{i=1}^d x_i y_i \cos \theta_i \right) \\ \times \prod_{i=1, i \neq r}^d j_{\alpha_i - \frac{1}{2}}(x_i y_i \cos \theta_i) \partial j_{\alpha_r - \frac{1}{2}}(x_r y_r \cos \theta_r) y_r \cos \theta_r (\sin \theta_i)^{2\alpha_i} d\theta.$$

We first estimate  $\mathcal{Q}_1(x, y)$ . By the left-hand-side identity in (8.23) and (8.28)

$$\frac{d}{dt} [L_n^\alpha(t) e^{-t/2}] = -(1/2)(L_n^\alpha(t) + 2L_{n-1}^{\alpha+1}(t)) e^{-t/2} = -(1/2)(L_n^{\alpha+1}(t) + L_{n-1}^{\alpha+1}(t)) e^{-t/2}.$$

Hence, by the definition of  $\mathcal{K}_n^\lambda$  in (8.4),

$$\partial \mathcal{K}_n^{k+|\alpha|+d}(t) = -[\mathcal{K}_n^{k+|\alpha|+d+1}(t) + \tilde{\mathcal{K}}_n^{k+|\alpha|+d+1}(t)]/2,$$

where  $\tilde{\mathcal{K}}_n^\lambda(t)$  is define as  $\mathcal{K}_n^\lambda(t)$  but with  $L_m^\lambda$  in the sum in (8.4) replaced by  $L_{m-1}^\lambda$ . Evidently,  $\tilde{\mathcal{K}}_n^\lambda(t)$  has the same properties as  $\mathcal{K}_n^\lambda(t)$ . Substituting the above in (8.37) and taking into account (8.7)-(8.8) we get

$$\mathcal{Q}_1(x, y) \leq cn^{(-k+|\alpha|+d+1)/2} \int_{[0, \pi]^d} \frac{|x_r - y_r \cos t_r| \prod_{i=1}^d t_i^{2\alpha_i} dt}{(\|x - y\|_2^2 + \sum_{i=1}^d x_i y_i t_i^2)^{(k+|\alpha|+d+1)/2}}.$$

Now, using the fact that

$$|x_r - y_r \cos t_r| \leq |x_r - y_r| + 2x_r y_r \sin^2(t_r/2) \leq |x_r - y_r| + x_r^{-1} (x_r y_r t_r^2)$$

and noticing that  $|x_r - y_r|$  can be cancelled by an  $1/2$  power of the main term in the denominator, whereas  $x_r y_r t_r^2$  needs a square of that much, we conclude that

$$\mathcal{Q}_1(x, y) \leq cn^{(-k+|\alpha|+d+1)/2} \int_{[0, \pi]^d} \frac{\prod_{i=1}^d t_i^{2\alpha_i} dt}{(\|x - y\|_2^2 + \sum_{i=1}^d x_i y_i t_i^2)^{(k+|\alpha|+d)/2}} \\ + cx_r^{-1} n^{(-k+|\alpha|+d+1)/2} \int_{[0, \pi]^d} \frac{\prod_{i=1}^d t_i^{2\alpha_i} dt}{(\|x - y\|_2^2 + \sum_{i=1}^d x_i y_i t_i^2)^{(k+|\alpha|+d-1)/2}}.$$

Both of the above integrals are of the form of  $M_n^{k, \alpha}$  defined in (8.9). In fact, we have

$$(8.39) \quad \mathcal{Q}_1(x, y) \leq cn^{1/2} M_n^{k, \alpha}(x, y) + cx_r^{-1} M_n^{k-1, \alpha}(x, y).$$

Furthermore, evidently  $|x_r - y_r \cos t_r| \leq |x_r - y_r| + x_r t_r^2$  and inserting  $t_r^2$  into the weight function of the integral, we obtain as above

$$(8.40) \quad \mathcal{Q}_1(x, y) \leq cn^{1/2} M_n^{k, \alpha}(x, y) + cx_r M_n^{k, \alpha+e_r}(x, y).$$

We next estimate  $\mathcal{Q}_2$ . Using the integral representation (8.6) for  $j_{\alpha-\frac{1}{2}}(x)$  we get

$$\partial j_{\alpha-\frac{1}{2}}(x) = c \int_{-1}^1 e^{ixt} t(1-t^2)^{\alpha-1} dt, \quad \alpha > 0,$$

while  $\partial j_{-\frac{1}{2}}(x) = c \sin x$ . Therefore,  $|\partial j_{\alpha-\frac{1}{2}}(x)| \leq c$  for  $\alpha \geq 0$ . Consequently, using also that  $y_r \leq cn^{1/2}$ , we obtain as in (8.9)

$$(8.41) \quad |\mathcal{Q}_2(x, y)| \leq cn^{1/2} M_n^{k, \alpha}(x, y).$$

Combining (8.39) and (8.41) gives

$$(8.42) \quad \left| \frac{\partial}{\partial x_r} \Lambda_n(x, y) \right| \leq cx_r^{-1} M_n^{k-1, \alpha}(x, y) + cn^{1/2} M_n^{k, \alpha}(x, y),$$

whereas combining (8.40) and (8.41) gives

$$(8.43) \quad \left| \frac{\partial}{\partial x_r} \Lambda_n(x, y) \right| \leq cn^{1/2} M_n^{k, \alpha}(x, y) + cx_r M_n^{k, \alpha+e_r}(x, y).$$

We are now in a position to establish estimate (3.11). Using (8.10) in (8.42) and combining the result with (8.32), we conclude that for  $x_r \geq n^{-1/2}$

$$(8.44) \quad \left| \frac{\partial}{\partial x_r} \Lambda_n(x, y) \right| \leq \frac{cn^{|\alpha|+d+1/2}}{(1+n^{1/2}\|x-y\|)^{k+|\alpha|+d-1}}.$$

On the other hand, using (8.10) in (8.43) and combining the result with (8.32) shows that estimate (8.44) holds for  $x_r \leq n^{-1/2}$  as well. Therefore, (8.44) holds for all  $x, y \in \mathbb{R}_+^d$ .

In going further, using (8.11) in (8.42) and combining the result with (8.35), we obtain for  $x_r \geq n^{-1/2}$  and  $1 \leq i \leq \ell$

$$(8.45) \quad \left| \frac{\partial}{\partial x_r} \Lambda_n(x, y) \right| \leq \frac{cn^{(d+1)/2}}{\prod_{i=1}^{\ell} (x_i y_i)^{\alpha_i + \frac{1}{2}} \prod_{i=\ell+1}^d (n^{-1})^{\alpha_i + \frac{1}{2}} (1+n^{1/2}\|x-y\|)^{k-|\alpha|-1}}.$$

On the other hand, using (8.11) in (8.43) and combining the result with (8.35), we see that the same bound (8.45) holds for  $x_r \leq n^{-1/2}$  as well. Therefore, (8.45) holds in general. Moreover, (8.45) holds for all possible permutations of the indices and combining it with (8.44) leads to

$$\left| \frac{\partial}{\partial x_r} \Lambda_n(x, y) \right| \leq \frac{cn^{(d+1)/2}}{\prod_{i=1}^d (x_i y_i + n^{-1})^{\alpha_i + \frac{1}{2}} (1+n^{1/2}\|x-y\|)^{k-|\alpha|-1}}.$$

Now, estimate (3.11) follows using (8.21) as before.

The proof of (3.13) is simple. By (8.22) and (8.24) it follows that

$$\left| \frac{d}{dx} \mathcal{F}_n^\alpha(x) \right| \leq cx^{-\alpha+1} e^{-\gamma x^2} \leq ce^{-\gamma' x^2} \quad \text{for } x \geq (6(1+v)n + 3\alpha + 3)^{1/2}.$$

This and (8.22) imply that if  $\max\{\|x\|^2, \|y\|^2\} \geq (6(1+v)n + 3\alpha + 3)^{1/2}$ , then

$$\left| \frac{\partial}{\partial x_r} \Lambda_n(x, y) \right| \leq cn^d e^{-\gamma'' \max\{\|x\|^2, \|y\|^2\}}, \quad \gamma'' > 0,$$

which yields (3.13).  $\square$

### 8.3. Proof of other localization estimates.

**Proof of Lemma 3.4.** We will derive estimate (3.15) from the following estimate: If  $s \in \mathbb{R}$ ,  $\gamma \geq 0$ ,  $\sigma > (2\gamma + 1)(|s| + 1) + 1$ , and  $z > 0$ , then

$$(8.46) \quad I := \int_0^\infty \frac{u^{2\gamma+1} du}{(1+u)^{(2\gamma+1)s}(1+|u-z|)^\sigma} \leq \frac{c}{(1+z)^{(2\gamma+1)(s-1)}}.$$

Consider first the case when  $s \geq 1$ . Then  $I = \int_0^{z/2} + \int_{z/2}^\infty =: J_1 + J_2$ . Evidently,

$$J_1 \leq (1+z)^{-\sigma} \int_0^z 1 du \leq c(1+z)^{-\sigma+1}$$

and

$$J_2 \leq \frac{c}{(1+z)^{(2\gamma+1)(s-1)}} \int_{z/2}^\infty \frac{du}{(1+|u-y|)^\sigma} \leq \frac{c}{(1+z)^{(2\gamma+1)(s-1)}} \quad (\sigma > 1).$$

Since  $\sigma > (2\gamma + 1)(s - 1) + 1$  the above estimates for  $J_1$  and  $J_2$  yield (8.46).

Let  $s < 1$ . Then we have

$$\begin{aligned} I &\leq \int_0^\infty \frac{(1+u)^{(2\gamma+1)(1-s)} du}{(1+|u-z|)^\sigma} = \int_{-z}^\infty \frac{(1+v+z)^{(2\gamma+1)(1-s)} du}{(1+|v|)^\sigma} \\ &\leq c \int_{-\infty}^\infty \frac{(1+|v|)^{(2\gamma+1)(1-s)} + z^{(2\gamma+1)(1-s)}}{(1+|v|)^\sigma} du \\ &\leq c \int_{-\infty}^\infty \frac{du}{(1+|v|)^{\sigma+(2\gamma+1)(s-1)}} + cz^{(2\gamma+1)(1-s)} \int_{-\infty}^\infty \frac{du}{(1+|v|)^\sigma} \\ &\leq \frac{c}{(1+z)^{(2\gamma+1)(1-s)}}. \end{aligned}$$

Here we used that  $\sigma > (2\gamma + 1)(1 - s) + 1$ . Therefore, (8.46) holds when  $s < 1$  as well.

We now proceed with the proof of (3.15). Denote by  $J$  the integral in (3.15). Using that  $|x_j - y_j| \leq \|x - y\|$ , we get

$$\begin{aligned} J &\leq \prod_{i=1}^d \int_0^\infty \frac{y_i^{2\alpha_i+1} dy_i}{(y_i + n^{-1/2})^{(2\alpha_i+1)s} (1 + n^{1/2}|x_i - y_i|)^{\sigma/d}} \\ &= n^{(2|\alpha|+d)s} \prod_{i=1}^d \int_0^\infty \frac{y_i^{2\alpha_i+1} dy_i}{(1 + n^{1/2}y_i)^{(2\alpha_i+1)s} (1 + |n^{1/2}x_i - n^{1/2}y_i|)^{\sigma/d}} \\ &= n^{(2|\alpha|+d)(s-1)-d/2} \prod_{i=1}^d \int_0^\infty \frac{u^{2\alpha_i+1} du}{(1+u)^{(2\alpha_i+1)s} (1+|u-n^{1/2}x_i|)^{\sigma/d}} \\ &\leq cn^{(2|\alpha|+d)(s-1)-d/2} \prod_{i=1}^d \frac{1}{(1+n^{1/2}x_i)^{(2\alpha_i+1)(s-1)}} = \frac{cn^{-d/2}}{W_\alpha(n; x)^{s-1}}. \end{aligned}$$

Here for the last inequality we used (8.46).  $\square$

**Proof of Theorems 3.7 and 3.8.** By (3.6) we have  $\mathcal{L}_\nu^\alpha(x) = 2^{-1/2} \mathcal{F}_\nu^\alpha(x^{1/2})x^{\alpha/2}$  and by and (3.2)  $\mathcal{M}_\nu^\alpha(x) = x^{\alpha+1/2} \mathcal{F}_\nu^\alpha(x)$  and hence

$$\tilde{\Lambda}_n(x, y) = 2^{-1} \Lambda_n(x^{1/2}, y^{1/2})x^{\alpha/2}y^{\alpha/2} \quad \text{and} \quad \Lambda_n^*(x, y) = \Lambda_n(x, y)x^{\alpha+1/2}y^{\alpha+1/2}$$

Now, it is easy to see that these relations and estimates (3.10) and (3.11) yield (3.18) and (3.19) as well as (3.20) and (3.21).  $\square$

**8.4. Proof of Lemma 3.6.** The main step is to prove Lemma 3.6 for dimension  $d = 1$ . To this end we will need a lemma which goes back to van der Corput (see e.g. [20, Vol. I, p. 197-198]).

**Lemma 8.1.** *If  $f''(u) \geq \rho > 0$  or  $f''(u) \leq -\rho < 0$  on  $[a, b]$ , then*

$$\left| \sum_{a \leq n \leq b} e^{2\pi i f(n)} \right| \leq (|f'(b) - f'(a)| + 2)(4\rho^{-1/2} + c).$$

Evidently, when  $d = 1$  Lemma 3.6 is immediate from the following lemma.

**Lemma 8.2.** *For any  $\varepsilon > 0$  and  $\delta > 0$  there exists a constant  $c > 0$  such that for  $n \geq 1/\varepsilon$*

$$(8.47) \quad \mathcal{A}_n(x) := e^{-x} \sum_{m=n}^{n+\lfloor \varepsilon n \rfloor} \frac{[L_m^\alpha(x)]^2}{L_m^\alpha(0)} \geq cn^{1/2} \left(x + \frac{1}{n}\right)^{-\alpha-1/2}, \quad 0 \leq x \leq (4-\delta)n.$$

**Proof.** We may assume that  $\varepsilon \leq 1$  and  $n \geq n_0$ , where  $n_0$  is sufficiently large. The proof uses the asymptotic of  $L_n^\alpha(x)$  and is divided into several cases.

*Case 1:*  $0 \leq x < c^\diamond n^{-1}$  with  $c^\diamond := (\alpha+1)(\alpha+3)$  ( $c^\diamond n^{-1}$  is larger than the smallest zero of  $L_n^\alpha$  [16, (6.31.12)]). We need the asymptotic formula [16, (8.22.4)-(8.22.5)]

$$e^{-x/2} x^{\alpha/2} L_n^\alpha(x) = N^{-\alpha} \frac{\Gamma(n+\alpha+1)}{n!} J_\alpha(2(Nx)^{1/2}) + x^{\alpha/2+2} \mathcal{O}(n^\alpha), \quad 0 < x \leq c/n,$$

where  $N = n + (\alpha+1)/2$ . Using also that  $J_\alpha(z) = \frac{z^\alpha}{2^\alpha \Gamma(\alpha+1)} + \mathcal{O}(z^{\alpha+2})$ , we obtain

$$e^{-x/2} L_n^\alpha(x) \sim n^\alpha + x^2 \mathcal{O}(n^\alpha) \geq cn^\alpha, \quad 0 \leq x < c/n.$$

Combining this with  $L_n^\alpha(0) = \binom{n+\alpha}{n} \sim n^\alpha$  we arrive at

$$\mathcal{A}_n(x) \geq c \sum_{m=n}^{n+\lfloor \varepsilon n \rfloor} m^\alpha \sim n^{\alpha+1}, \quad 0 \leq x < c^\diamond n^{-1},$$

which proves (8.47) in this case.

*Case 2:*  $c^\diamond n^{-1} \leq x \leq c_* n^{-1}$ , where the constant  $c_* > 1$  will be selected later on. In this case we use relation (2.16) and (2.19) to conclude that

$$e^{-x} |L_n^\alpha(x)|^2 \sim n^{2\alpha+2} (x - t_{k_x, n})^2.$$

Furthermore, by a theorem of Tricomi (see [8] for the references), we know that for all the zeros of  $L_n^\alpha$  in the interval  $0 < x < c/n$  we have  $t_{k, n} = \frac{j_{\alpha, k}^2}{n} (1 + \mathcal{O}(n^{-2}))$  as  $n \rightarrow \infty$ , where  $j_{\alpha, k}$ ,  $k = 1, 2, \dots$ , are the positive zeros, in increasing order, of the Bessel function  $J_\alpha(x)$ . Consequently,

$$\begin{aligned} \mathcal{A}_n(x) &\geq cn^\alpha \sum_{m=n}^{n+\lfloor \varepsilon n \rfloor} ((mx - j_{\alpha, k_x}^2)^2 - cm^{-1} |mx - j_{\alpha, k_x}|) \\ &\geq cn^\alpha \left( \sum_{m=n}^{n+\lfloor \varepsilon n \rfloor} (mx - j_{\alpha, k_x}^2)^2 - c \right) \geq cn^{\alpha+1}. \end{aligned}$$

Here for the last estimate we used that  $j_{\alpha, k} \rightarrow \infty$  as  $k \rightarrow \infty$  and hence there are only finitely many zeros of  $J_\alpha(x)$  such that  $j_{\alpha, k}^2 \leq c_* n^{-1} (n + \lfloor \varepsilon n \rfloor) \leq c$ ; the argument is the same as in the analogous situation for Jacobi polynomials in [9].

*Case 3:*  $c_* n^{-1} \leq x \leq c^*$ , where  $c_*$  is sufficiently large and its value is to be determined. In this case we use the asymptotic formula for  $L_n^\alpha(x)$  [16, (8.22.6)]:

$$e^{-x/2} L_n^\alpha(x) = \pi^{-1/2} x^{-\alpha/2-1/4} n^{\alpha/2-1/4} \times \left[ \cos\left(2(nx)^{1/2} - \alpha\pi/2 - \pi/4\right) + \mathcal{O}(1)(nx)^{-1/2} \right],$$

which holds for  $c'n^{-1} \leq x \leq c''$  and  $\mathcal{O}(1)$  depends only on  $c', c''$ . We denote  $\gamma := \alpha\pi/2 + \pi/4$  and deduce from above

$$\begin{aligned} x^{\alpha+1/2} \mathcal{A}_n(x) &\geq cn^{-\alpha} e^{-x} x^{\alpha+1/2} \sum_{m=n}^{n+[\varepsilon n]} [L_m^\alpha(x)]^2 \\ &\geq c \sum_{m=n}^{n+[\varepsilon n]} m^{-1/2} \left( \cos[2(mx)^{1/2} - \gamma] + \mathcal{O}(1)(nx)^{-1/2} \right)^2 \\ &\geq cn^{-1/2} \sum_{m=n}^{n+[\varepsilon n]} \cos^2[2(mx)^{1/2} - \gamma] + \mathcal{O}(1)c_*^{-1/2} n^{1/2}. \end{aligned}$$

Using the fact that  $2\cos^2 t = 1 + \cos 2t$  and then write  $2\cos 2t = e^{2it} + e^{-2it}$ , we see that

$$\Sigma := 4 \sum_{m=n}^{n+[\varepsilon n]} \cos^2[2(mx)^{1/2} - \gamma] \geq 2[\varepsilon n] + \sum_{m=n}^{n+[\varepsilon n]} \left[ e^{2\pi i(y\sqrt{m}-\gamma')} + e^{-2\pi i(y\sqrt{m}-\gamma')} \right],$$

where  $y := (2/\pi)\sqrt{x}$  and  $\gamma' := 2\gamma/\pi$ . The last sum can be estimated by making use of Lemma 8.1 with  $f(u) = y\sqrt{u}$ ,  $a = n$  and  $b = n + [\varepsilon n]$ . We get

$$\begin{aligned} \Sigma &\geq 2[\varepsilon n] - 2(2 + x^{1/2} n^{-1/2})(c + 24x^{-1/4} n^{3/4}) \\ &\geq 2[\varepsilon n] - 2(2 + (c^*)^{1/2} n^{-1/2})(c + 24c_*^{-1/4} n). \end{aligned}$$

Putting the above estimates together, we arrive at

$$x^{\alpha+1/2} \mathcal{A}_n(x) \geq cn^{-1/2} \left( 2[\varepsilon n] - 2(2 + (c^*)^{1/2} n^{-1/2})(c + 24c_*^{-1/4} n) \right) + \mathcal{O}(1)c_*^{-1/2} n^{1/2}.$$

Choosing  $c_*$  sufficiently large shows that the right-hand side of the above inequality is bounded below by  $cn^{1/2}$  for sufficiently large  $n$ . Thus (8.47) is proved in this case.

*Case 4:*  $c^* \leq x \leq (4-\delta)n$ . Here we apply another asymptotic formula of Laguerre polynomial [16, (8.22.9)]: For  $x = (4m + 2\alpha + 2) \cos^2 \phi$  with  $\varepsilon \leq \phi \leq \pi/2 - \varepsilon m^{-1/2}$ ,  $x^{\alpha/2+1/4} e^{-x/2} L_m^\alpha(x) = (-1)^m (\pi \sin \phi)^{-1/2} m^{\alpha/2-1/4}$

$$\times \left\{ \sin \left[ \left(m + \frac{\alpha+1}{2}\right) (\sin 2\phi - 2\phi) + 3\pi/4 \right] + \mathcal{O}(1)(mx)^{-1/2} \right\}.$$

Note that the range of  $x$  above covers the range of this case. From above, as in Case 3, we obtain

$$\begin{aligned} x^{\alpha+1/2} \mathcal{A}_n(x) &\geq cn^{-\alpha} e^{-x} x^{\alpha+1/2} \sum_{m=n}^{n+[\varepsilon n]} [L_m^\alpha(x)]^2 \\ &\geq cn^{-1/2} \sum_{m=n}^{n+[\varepsilon n]} \sin^2 \left[ \left(m + \frac{\alpha+1}{2}\right) (\sin 2\phi - 2\phi) + 3\pi/4 \right] + \mathcal{O}(1)(c^*)^{-1/2}. \end{aligned}$$



The last sum is again bounded below by  $cn$ , which can be proved either by using Lemma 8.1 or by summing up using simple trigonometric identities. This shows again that (8.47) holds.  $\square$

**Proof of (3.16) in the case  $d \geq 2$ .** We may again assume  $\varepsilon \leq 1$ . We will use induction on  $d$ . To indicate the dependence of  $\mathcal{F}_m^\alpha$  on  $d$  we write  $\mathcal{F}_{m,d}^\alpha := \mathcal{F}_m^\alpha$ . Assume that (3.17) has been established for dimensions up to  $d-1$ . By definition

$$\mathcal{F}_{m,d}^\alpha(x, x) = \sum_{k=0}^m \left[ \mathcal{F}_k^{\alpha_d}(x_d) \right]^2 \mathcal{F}_{m-k, d-1}^{\alpha'}(x', x'), \quad x = (x', x_d), \quad \alpha = (\alpha', \alpha_d),$$

and hence

$$\begin{aligned} \sum_{m=n}^{n+[d\varepsilon n]} \mathcal{F}_{m,d}^\alpha(x, x) &\geq \sum_{m=n}^{n+[d\varepsilon n]} \sum_{k=0}^{[ \varepsilon n ]} \left[ \mathcal{F}_k^{\alpha_d}(x_d) \right]^2 \mathcal{F}_{m-k, d-1}^{\alpha'}(x', x') \\ (8.48) \qquad &= \sum_{k=0}^{[ \varepsilon n ]} \left[ \mathcal{F}_k^{\alpha_d}(x_d) \right]^2 \sum_{m=n}^{n+[d\varepsilon n]} \mathcal{F}_{m-k, d-1}^{\alpha'}(x', x') \\ &\geq \sum_{k=0}^{[ \varepsilon n ]} \left[ \mathcal{F}_k^{\alpha_d}(x_d) \right]^2 \sum_{j=n}^{n+[(d-1)\varepsilon n]} \mathcal{F}_{j, d-1}^{\alpha'}(x', x'). \end{aligned}$$

It follows by (3.1) and (2.12)-(2.14) that for  $0 \leq x \leq \sqrt{(4-\delta)n}$

$$\sum_{k=0}^n \left[ \mathcal{F}_k^\alpha(x) \right]^2 = ce^{-x^2} K_n^\alpha(x^2, x^2) \geq cn^{1/2} \left( x^2 + \frac{1}{n} \right)^{-\alpha-1/2} \geq cn^{1/2} (x+n^{-1/2})^{-2\alpha-1}.$$

Combining this estimate with (8.48) and the inductive assumption shows that (3.17) holds in dimension  $d$ .  $\square$

**Proof of Proposition 4.1.** We first prove (4.2). Let  $g \in V_n$ . Assume  $1 < q < \infty$  and let  $\Lambda_n$  be the kernel from (3.7), with  $\hat{a}$  admissible of type  $(a)$ . Evidently  $g = \Lambda_n * g$  and using Hölder's inequality and Proposition 3.3 we obtain for  $x \in \mathbb{R}_+^d$

$$\begin{aligned} |g(x)| &\leq \|W_\alpha(n; \cdot)^{s+\frac{1}{p}-\frac{1}{q}} g(\cdot)\|_q \left( \int_{\mathbb{R}_+^d} \left| \Lambda_n(x, y) W_\alpha(n; y)^{-s-\frac{1}{p}+\frac{1}{q}} \right|^{q'} w_\alpha(y) dy \right)^{\frac{1}{q'}} \\ &\leq c \frac{n^{d/2}}{W_\alpha(n; x)^{1/2}} \left( \int_{\mathbb{R}_+^d} \frac{w_\alpha(y) dy}{W_\alpha(n; y)^{\frac{q'}{2}+\beta} (1+n^{1/2}\|x-y\|)^\sigma} \right)^{\frac{1}{q'}} \|W_\alpha(n; \cdot)^{s+\frac{1}{p}-\frac{1}{q}} g(\cdot)\|_q, \end{aligned}$$

where  $\beta := q'(s + \frac{1}{p} - \frac{1}{q})$ . To estimate the last integral we use estimate (3.15) from Lemma 3.4 to obtain

$$(8.49) \qquad |g(x)| \leq c \frac{n^{d/2q}}{W_\alpha(n; x)^{s+1/p}} \|W_\alpha(n; \cdot)^{s+\frac{1}{p}-\frac{1}{q}} g(\cdot)\|_q$$

and hence

$$(8.50) \qquad \|W_\alpha(n; \cdot)^{s+\frac{1}{p}} g(\cdot)\|_\infty \leq cn^{d/2q} \|W_\alpha(n; \cdot)^{s+\frac{1}{p}-\frac{1}{q}} g(\cdot)\|_q, \quad 1 < q \leq \infty.$$

If  $0 < q \leq 1$ , then the above estimate with  $q = 2$  gives

$$\begin{aligned} \|W_\alpha(n; \cdot)^{s+\frac{1}{p}} g(\cdot)\|_\infty &\leq cn^{d/4} \|W_\alpha(n; \cdot)^{s+\frac{1}{p}-\frac{1}{2}} g(\cdot)\|_2 \\ &\leq cn^{d/4} \|W_\alpha(n; \cdot)^{s+1/p} g(\cdot)\|_\infty^{1-q/2} \|W_\alpha(n; \cdot)^{s+\frac{1}{p}-\frac{1}{q}} g(\cdot)\|_q^{q/2}. \end{aligned}$$

Consequently, (8.50) holds for  $0 < q \leq 1$  as well.

Let  $0 < q < p < \infty$ . Using (8.50), we have

$$\begin{aligned} & \|W_\alpha(n; \cdot)^s g(\cdot)\|_p \\ &= \left( \int_{\mathbb{R}_+^d} \left| W_\alpha(n; x)^{s+\frac{1}{p}} g(x) \right|^{p-q} \left| W_\alpha(n; x)^{s+\frac{1}{p}-\frac{1}{q}} g(x) \right|^q w_\alpha(x) dx \right)^{1/p} \\ &\leq \|W_\alpha(n; \cdot)^{s+\frac{1}{p}} g(\cdot)\|_\infty^{1-q/p} \|W_\alpha(n; \cdot)^{s+\frac{1}{p}-\frac{1}{q}} g(\cdot)\|_q^{q/p} \\ &= cn^{(d/2)(1/q-1/p)} \|W_\alpha(n; \cdot)^{s+\frac{1}{p}-\frac{1}{q}} g(\cdot)\|_q. \end{aligned}$$

Hence (4.2) holds when  $p < \infty$ . In the case  $p = \infty$  (4.2) follows from (8.50).

To prove (4.1) we first assume that  $1 < q < \infty$ . We use again that  $g = \Lambda_n * g$ , Hölder's inequality, Proposition 3.3, and that  $W_\alpha(n; x) \geq n^{-|\alpha|-d/2}$  to obtain

$$|g(x)| \leq c \|g\|_q \left( n^{d/2} W_\alpha(n; x)^{-1} \right)^{1/q} \leq cn^{(d+|\alpha|)/q} \|g\|_q, \quad x \in \mathbb{R}_+^d,$$

and hence  $\|g\|_\infty \leq cn^{(d+|\alpha|)/q} \|g\|_q$ . For the rest of the proof of (4.1) one proceeds similarly as in the proof of (4.2). We skip the details.

To prove estimate (4.3) we first observe that (8.49) with  $s = \gamma + 1/p - 1/q$  yields

$$|g(x)| \leq c \frac{n^{d/2q}}{W_\alpha(n; x)^{s+1/q}} \|W_\alpha(n; \cdot)^s g(\cdot)\|_q, \quad 1 < q < \infty,$$

and, since  $W_\alpha(n; x) \geq n^{-|\alpha|-\frac{d}{2}}$ , we get  $\|g\|_\infty \leq cn^{(|\alpha|+\frac{d}{2})s+(|\alpha|+d)/q} \|W_\alpha(n; \cdot)^s g(\cdot)\|_q$ . The remaining part of the proof is similar to the proof of (4.2). We omit it.  $\square$

**Proof of Lemma 4.4.** (a) By (2.10) and the definition of  $\mathcal{F}_n^\alpha$ , it follows that  $\|\mathcal{F}_n^\alpha\|_\infty \leq cn^{\alpha/2}$ . Hence, using (8.27) if  $|x| \leq 1$  and (8.30) if  $|x| \geq 1$ , we obtain

$$\left| \frac{d}{dx} \mathcal{F}_n^\alpha(x) \right| \leq cn^{(\alpha+1)/2}, \quad x \in \mathbb{R}_+.$$

Furthermore, taking one more derivative of (8.24) and using (8.27) shows that

$$\begin{aligned} \frac{d^2}{dx^2} \mathcal{F}_n^\alpha(x) &= -[\mathcal{F}_n^\alpha(x) + 2\sqrt{n}\mathcal{F}_{n-1}^{\alpha+1}(x)] + x \frac{d}{dx} \mathcal{F}_n^\alpha(x) + 2\sqrt{n}x \frac{d}{dx} \mathcal{F}_{n-1}^{\alpha+1}(x) \\ &= -[\mathcal{F}_n^\alpha(x) + 2\sqrt{n}\mathcal{F}_{n-1}^{\alpha+1}(x)] \\ &\quad - (\alpha+1)\mathcal{F}_n^\alpha(x) + b_{n+1}\mathcal{F}_{n+1}^\alpha(x) - b_n\mathcal{F}_{n-1}^\alpha(x) \\ &\quad + 2\sqrt{n}[-(\alpha+1)\mathcal{F}_{n-1}^{\alpha+1}(x) + b_n\mathcal{F}_n^{\alpha+1}(x) - b_{n-1}\mathcal{F}_{n-2}^{\alpha+1}(x)], \end{aligned}$$

which allows us to iterate and express  $\frac{d^{k+1}}{dx^{k+1}} \mathcal{F}_n^\alpha(x)$  in terms of  $\frac{d^{k-1}}{dx^{k-1}} \mathcal{F}_n^\alpha(x)$  and  $\frac{d^{k-1}}{dx^{k-1}} \mathcal{F}_n^{\alpha+1}(x)$ . The recurrence relation (8.29) allows us to use induction to conclude that

$$\left| \frac{d^k}{dx^k} \mathcal{F}_n^\alpha(x) \right| \leq cn^{(\alpha+k)/2}, \quad x \in \mathbb{R}_+.$$

Therefore, for the product Laguerre functions, we have

$$|\partial^\beta \mathcal{F}_\nu^\alpha(x)| \leq c(|\nu|+1)^{(|\alpha|+|\beta|)/2}, \quad |\nu| = n, \quad \beta \in \mathbb{N}_0^d, \quad x \in \mathbb{R}_+^d.$$

Furthermore, together with the three term relation (8.26), the above inequality also shows that

$$|x^{2\gamma} \partial^\beta \mathcal{F}_\nu^\alpha(x)| \leq c(|\nu|+1)^{(|\alpha|+|\beta|+2|\gamma|)/2}, \quad |\nu| = n, \quad \beta, \gamma \in \mathbb{N}_0^d, \quad x \in \mathbb{R}_+^d.$$

Hence, if  $|\langle \phi, \mathcal{F}_\nu^\alpha \rangle| \leq c_k (|\nu| + 1)^{-k}$  for all  $k$ , then

$$x^\gamma \partial^\beta \phi(x) = \sum_{\nu \in \mathbb{N}_0^d} \langle \phi, \mathcal{F}_\nu^\alpha \rangle x^\gamma \partial^\beta \mathcal{F}_\nu^\alpha(x),$$

where the series converges uniformly and hence

$$(8.51) \quad |x^\gamma \partial^\beta \phi(x)| \leq c \sum_{\nu \in \mathbb{N}_0^d} |\langle \phi, \mathcal{F}_\nu^\alpha \rangle| (|\nu| + 1)^{(|\alpha| + |\beta| + 2|\gamma|)/2} \leq c_k P_k^*(\phi)$$

if  $k > d + |\alpha| + |\beta| + 2|\gamma|/2$ , which shows that  $\phi \in \mathcal{S}_+$ .

(b) Assuming that  $\phi \in \mathcal{S}_+$  we next show that  $|\langle \phi, \mathcal{F}_\nu^\alpha \rangle|$  has the claimed decay. From the well known second order differential equation satisfied by  $L_n^\alpha$ , a straightforward computation shows that  $\mathcal{F}_n^\alpha(x)$  satisfies the equation

$$y'' + \frac{2\alpha + 1}{x} y' - x^2 y + 2(2n + \alpha + 1)y = 0.$$

In particular, it follows that  $\mathcal{F}_\nu^\alpha(x)$  satisfies, for each  $i = 1, 2, \dots, d$ , the equation

$$(8.52) \quad \mathcal{D}_{x_i} u + x_i^2 u = 2(2\nu_i + \alpha_i + 1)u, \quad \text{where } \mathcal{D}_{x_i} := -\partial_i^2 - (2\alpha_i + 1)x_i^{-1}\partial_i$$

and  $\partial_i = \frac{\partial}{\partial x_i}$ .

Let  $k \geq 1$  and assume that the multi-index  $\nu$  is fixed and  $\|\nu\| = \max_{1 \leq j \leq d} \nu_j \geq k$ . Choose  $i$  so that  $\nu_i = \|\nu\|$  and denote  $\hat{x}_i = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_d)$ . Denote briefly  $\mathcal{U}_r(x) := \partial_i^r(\phi(x)e^{x_i^2/2})$ . Then by Taylor's identity

$$\phi(x)e^{x_i^2/2} - \sum_{r=0}^{2k-1} x_i^r \mathcal{U}_r(\hat{x})/r! = \frac{x_i^{2k}}{(2k-1)!} \int_0^1 (1-t)^{2k-1} \mathcal{U}_{2k}(\hat{x} + tx_i e_i) dt,$$

which easily leads to

$$(8.53) \quad \begin{aligned} \phi_i(x) &:= \phi(x) - e^{-x_i^2/2} \sum_{r=0}^{2k-1} x_i^r \mathcal{U}_r(\hat{x})/r! \\ &= x_i^{2k} \int_0^1 (1-t)^{2k-1} \sum_{j=0}^{2k} b_{2k-j}(tx_i) \partial_i^j \phi(\hat{x} + tx_i e_i) e^{-x_i^2(1-t^2)} dt, \end{aligned}$$

where  $b_j(\cdot)$  ( $0 \leq j \leq 2k$ ) is a polynomial of degree  $\leq j$  and  $e_i$  is the  $i$ th coordinate vector in  $\mathbb{R}^d$ . Then by the orthogonality of  $\mathcal{F}_\nu^{\alpha_i}$  (recall that  $\nu_i \geq 2k$ ) and (8.52) it follows that

$$\langle \phi, \mathcal{F}_\nu^\alpha \rangle = \langle \phi_i, \mathcal{F}_\nu^\alpha \rangle = \frac{1}{2(2\nu_i + \alpha_i + 1)} \langle \phi_i, (\mathcal{D}_{x_i} + x_i^2) \mathcal{F}_\nu^\alpha \rangle.$$

The operator  $\mathcal{D}_{x_i}$  can be written in a self-adjoint form  $x_i^{2\alpha_i+1} \mathcal{D}_{x_i} = \partial_i (x_i^{2\alpha_i+1} \partial_i)$ . We use this and integration by parts to obtain

$$\begin{aligned} \langle \phi_i, \mathcal{D}_{x_i} \mathcal{F}_\nu^\alpha \rangle &= \int_{\mathbb{R}_+^{d-1}} \int_{\mathbb{R}_+} \phi_i(x) \partial_i (x_i^{2\alpha_i+1} \partial_i \mathcal{F}_\nu^\alpha(x)) dx_i d\hat{x} \\ &= \int_{\mathbb{R}_+^{d-1}} \int_{\mathbb{R}_+} \partial_i (x_i^{2\alpha_i+1} \partial_i \phi_i(x)) \mathcal{F}_\nu^\alpha(x) dx_i d\hat{x} = \langle \mathcal{D}_{x_i} \phi_i, \mathcal{F}_\nu^\alpha \rangle. \end{aligned}$$

Consequently,

$$(8.54) \quad \begin{aligned} \langle \phi, \mathcal{F}_\nu^\alpha \rangle &= \frac{1}{2(2\nu_i + \alpha_i + 1)} \langle (\mathcal{D}_{x_i} + x_i^2) \phi_i, \mathcal{F}_\nu^\alpha \rangle \\ &= \frac{1}{2^k (2\nu_i + \alpha_i + 1)^k} \langle (\mathcal{D}_{x_i} + x_i^2)^k \phi_i, \mathcal{F}_\nu^\alpha \rangle, \end{aligned}$$

where we iterated  $k$  times. It is easy to see that there is a representation of the form

$$(\mathcal{D}_{x_i} + x_i^2)^k = (-\partial_i^2 - (2\alpha_i + 1)x_i^{-1}\partial_i + x_i^2)^k = \sum_{j=0}^{2k} \sum_{\ell=-2k}^{2k-j} a_{j\ell} x_i^{-\ell} \partial_i^j$$

for some constants  $a_{j\ell}$ . On the other hand, by (8.53) it follows that if  $j + \ell \leq 2k$

$$\sup_x |x_i^{-\ell} \partial_i^j \phi_i(x)| \leq c \max_{|\gamma| \leq 4k, |\beta| \leq 2k+j} \sup_x |x^\gamma \partial^\beta \phi(x)| = c \max_{|\gamma| \leq 4k, |\beta| \leq 2k+j} P_{\beta, \gamma}(\phi).$$

We use the above in (8.54) to obtain

$$(8.55) \quad \begin{aligned} |\langle \phi, \mathcal{F}_\nu^\alpha \rangle| &\leq \frac{1}{2^k (2\nu_i + \alpha_i + 1)^k} \max_{|\gamma| \leq 4k, |\beta| \leq 4k} P_{\beta, \gamma}(\phi) \|\mathcal{F}_\nu^\alpha\|_1 \\ &\leq c |\nu|^{-k + (|\alpha| + d)/2} \max_{|\gamma| \leq 4k, |\beta| \leq 4k} P_{\beta, \gamma}(\phi), \quad \|\nu\| \geq k. \end{aligned}$$

Here we also used that  $\|\mathcal{F}_\nu^\alpha\|_1 \leq c |\nu|^{(|\alpha| + d)/2}$  which follows from Lemma 2.1. Estimate (8.55) shows that  $|\langle \phi, \mathcal{F}_\nu^\alpha \rangle| \leq c_k (|\nu| + 1)^{-k + (|\alpha| + d)/2}$  for any  $k \geq 1$ . Thus  $|\langle \phi, \mathcal{F}_\nu^\alpha \rangle|$  has the claimed decay.

The equivalence of the topologies on  $\mathcal{S}_+$  induced by the semi-norms  $P_{\gamma, \beta}$  from (4.7) and the norms  $P_k^*$  from (4.11) follows readily by (8.51) and (8.55).  $\square$

## 9. PROOFS FOR SECTIONS 6-7

**Proof of Proposition 6.4.** We shall use a standard decomposition of unity argument. Suppose  $\widehat{b} \in C^\infty(\mathbb{R})$  satisfies the conditions:  $\text{supp } \widehat{b} \subset [1/4, 4]$ ,  $b \geq 0$ , and  $\widehat{b}(t) + \widehat{b}(4t) = 1$  on  $[1/4, 1]$ ; hence  $\sum_{\ell=0}^{\infty} \widehat{b}(4^{-\ell}t) = 1$ ,  $t \in [1, \infty)$ . Now, define

$$\Phi_0(x, y) := m(0) \mathcal{F}_0^\alpha(x, y) \quad \text{and} \quad \Phi_\ell(x, y) := \sum_{j=0}^{4^\ell} m(j) \widehat{b}(j/4^{\ell-1}) \mathcal{F}_j^\alpha(x, y), \quad \ell \geq 1.$$

Then for the kernel  $K(x, y)$  of the operator  $T_m^\alpha$  we have  $K(x, y) = \sum_{\ell=0}^{\infty} \Phi_\ell(x, y)$ . By (6.6) it readily follows that  $\|(d/dt)^k [m(t) \widehat{b}(t/4^{\ell-1})]\|_\infty \leq c 4^{-\ell k}$  and just as in the proof of Theorem 3.2 (using also (5.17)) we get for  $x, y \in \mathbb{R}_+^d$

$$|\Phi_\ell(x, y)| \leq \frac{c 2^{\ell d}}{W(4^\ell; y) (1 + 2^\ell \|x - y\|)^\sigma}, \quad \left| \frac{\partial}{\partial y_r} \Phi_\ell(x, y) \right| \leq \frac{c 2^{\ell(d+1)}}{W(4^\ell; y) (1 + 2^\ell \|x - y\|)^\sigma},$$

for  $1 \leq r \leq d$ , where  $\sigma = k - (5/2)|\alpha| - (3/4)d - 2$ . By a simple standard argument these two estimates ( $\sigma > d + 1$ ) lead to

$$|K(x, y)| \leq \frac{c}{w_\alpha(y) \|x - y\|^d} \quad \text{and} \quad \left| \frac{\partial}{\partial y_r} K(x, y) \right| \leq \frac{c}{w_\alpha(y) \|x - y\|^{d+1}}, \quad 1 \leq r \leq d.$$

As in the weighted case on  $\mathbb{R}^d$  (see [15]), these estimates show that  $T_m^\alpha$  is a Calderón-Zygmund type operator and hence  $T_m^\alpha$  is bounded on  $L^p(w_\alpha)$ ,  $1 < p < \infty$ .  $\square$

**Proof of Lemma 6.8.** Using the orthogonality of Laguerre functions, we have  $\Phi_j * \psi_\xi(x) = 0$  for  $\xi \in \mathcal{X}_m$  if  $|m - j| \geq 2$ .

Let  $\xi \in \mathcal{X}_m$ ,  $j - 1 \leq m \leq j + 1$ . Assume first that  $\|\xi\| \leq (1 + \delta)\sqrt{6} \cdot 2^m$ . From (5.26)-(5.27) it follows that

$$\begin{aligned} |\Phi_j * \psi_\xi(x)| &\leq c_\sigma \frac{2^{m3d/2}}{\sqrt{W_\alpha(4^m; x)}} \int_{\mathbb{R}_+^d} \frac{w_\alpha(y)}{W_\alpha(4^m; y)(1 + 2^m\|x - y\|)^\sigma(1 + 2^m\|y - \xi\|)^\sigma} dy \\ &\leq \frac{c2^{m3d/2}}{\sqrt{W_\alpha(4^m; x)}} \int_{\mathbb{R}^d} \frac{dy}{(1 + 2^m\|x - y\|)^\sigma(1 + 2^m\|y - \xi\|)^\sigma} \quad (\sigma > d) \\ &\leq \frac{c2^{md/2}}{\sqrt{W_\alpha(4^m; x)}(1 + 2^m\|x - \xi\|)^\sigma} \leq \frac{c2^{md/2}}{\sqrt{W_\alpha(4^m; \xi)}(1 + 2^m\|x - \xi\|)^{\sigma - 2|\alpha| - 2d}} \\ &\leq \frac{c}{\mu(R_\xi)^{1/2}(1 + 2^m\|x - \xi\|)^{\sigma - 2|\alpha| - 2d}}, \end{aligned}$$

where for the last two inequalities we used (5.14)-(5.17). Since  $\sigma$  can be arbitrarily large the claimed estimate (6.13) follows.

Let  $\|\xi\| > (1 + \delta)\sqrt{6} \cdot 2^m$ . Just as above we use (5.26) and (5.28) to obtain

$$\begin{aligned} |\Phi_j * \psi_\xi(x)| &\leq c_\sigma \frac{2^{m(d-L)}}{\sqrt{W_\alpha(4^m; x)}} \int_{\mathbb{R}_+^d} \frac{w_\alpha(y)}{W_\alpha(4^m; y)(1 + 2^m\|x - y\|)^\sigma(1 + 2^m\|y - \xi\|)^\sigma} dy \\ &\leq \frac{c2^{m(d-L)}}{\sqrt{W_\alpha(4^m; \xi)}(1 + 2^m\|x - \xi\|)^{\sigma - 2|\alpha| - 2d}}. \end{aligned}$$

Since, in general,  $\mu(R_\xi) \leq c2^{-md/3}W_\alpha(4^m; \xi)$  and  $L$  can be arbitrarily large the above again leads to (6.13).  $\square$

**Proof of Lemma 6.10.** Denote

$$(9.1) \quad h_j^\star(x) := \sum_{\eta \in \mathcal{X}_j} \frac{|h_\eta|}{(1 + 2^j d(x, R_\eta))^\kappa}, \quad \kappa := \lambda - (2|\alpha| + d)|\rho|/d,$$

where  $d(x, E) := \inf_{y \in E} \|x - y\|$  is the  $\ell^\infty$  distance of  $x$  from  $E$ . We will show that

$$(9.2) \quad h_j^\star(x) \leq c\mathcal{M}_t \left( \sum_{\omega \in \mathcal{X}_j} |h_\omega| \mathbb{1}_{R_\omega} \right)(x), \quad x \in \mathbb{R}_+^d.$$

Evidently,  $h_j^*(x) \leq h_j^\star(x)$ ,  $x \in \mathbb{R}_+^d$ , and hence (9.2) implies (6.16). On the other hand, using (5.17) we have for  $\xi \in \mathcal{X}_j$

$$W_\alpha(4^j; \xi)^{-\rho/d} h_\xi^* \leq \sum_{\eta \in \mathcal{X}_j} \frac{W_\alpha(4^j; \eta)^{-\rho/d} |h_\eta|}{(1 + 2^j \|\xi - \eta\|)^{\lambda - (2|\alpha| + d)|\rho|/d}} \leq cH_j^\star(x) \quad \text{for } x \in R_\xi$$

where  $H_\eta := W_\alpha(4^j; \eta)^{-\rho/d} h_\eta$ . Therefore, (9.2) yields (6.17) as well.

By the definition of  $Q_j$  in (5.12) it follows that there exists a constant  $c_\star > 0$  depending only on  $d$  such that

$$Q_j := \cup_{\xi \in \mathcal{X}_j} R_\xi \subset [0, c_\star 2^j]^d.$$

Let  $x \in \mathbb{R}^d$ . To prove (9.2) we consider two cases for  $x$ .

*Case 1:*  $\|x\| > 2c_*2^j$ . Then  $d(x, R_\eta) > \|x\|/2$  for  $\eta \in \mathcal{X}_j$  and hence

$$(9.3) \quad \begin{aligned} h_j^\star(x) &= \sum_{\eta \in \mathcal{X}_j} \frac{|h_\eta|}{(1 + 2^j d(x, R_\eta))^\kappa} \leq \frac{c}{(2^j \|x\|)^\kappa} \sum_{\eta \in \mathcal{X}_j} |h_\eta| \\ &\leq \frac{c4^{jd\varrho}}{(2^j \|x\|)^\kappa} \left( \sum_{\eta \in \mathcal{X}_j} |h_\eta|^t \right)^{1/t}, \end{aligned}$$

where  $\varrho := 1 - \min\{1, 1/t\} \leq 1$  and for the last estimate we use Hölder's inequality if  $t > 1$  and the  $t$ -triangle inequality if  $t < 1$ .

Denote  $Q_x := [0, \|x\|]^d$ . Evidently,  $\mu(Q_x) \sim \|x\|^{2(|\alpha|+d)}$  and combining this with (9.3) we arrive at

$$\begin{aligned} h_j^\star(x) &\leq \frac{c4^{jd}\|x\|^{2(|\alpha|+d)/t}}{(2^j \|x\|)^\kappa} \left( \frac{1}{\mu(Q_x)} \int_{Q_x} \left( \sum_{\eta \in \mathcal{X}_j} |h_\eta| \mathbb{1}_{R_\eta}(y) \right)^t w_\alpha(y) dy \right)^{1/t} \\ &\leq c2^{j(2d-\kappa)} \|x\|^{2(|\alpha|+d)/t-\kappa} \mathcal{M}_t \left( \sum_{\eta \in \mathcal{X}_j} |h_\eta| \mathbb{1}_{R_\eta} \right)(x) \leq c \mathcal{M}_t \left( \sum_{\eta \in \mathcal{X}_j} |h_\eta| \mathbb{1}_{R_\eta} \right)(x) \end{aligned}$$

as claimed. Here we used the fact that  $\kappa > \max\{2d, 2(|\alpha| + d)/t\}$ .

*Case 2:*  $\|x\| \leq 2c_*2^j$ . We first subdivide the tiles  $\{R_\eta\}_{\eta \in \mathcal{X}_j}$  into boxes of almost equal sides of length  $\sim 2^{-j}$ . By the construction of the tiles (see (5.11)) there exists a constant  $\tilde{c} > 0$  such that the minimum side of each tile  $R_\eta$  is  $\geq \tilde{c}2^{-j}$ . Now, evidently each tile  $R_\eta$  can be subdivided into a disjoint union of boxes  $R_\theta$  with centers  $\theta$  such that

$$\theta + [-\tilde{c}2^{-j-1}, \tilde{c}2^{-j-1}]^d \subset R_\theta \subset \theta + [-\tilde{c}2^{-j}, \tilde{c}2^{-j}]^d.$$

Denote by  $\tilde{\mathcal{X}}_j$  the set of centers of all boxes obtained by subdividing the tiles from  $\mathcal{X}_j$ . Also, set  $h_\theta := h_\eta$  if  $R_\theta \subset R_\eta$ . Evidently,

$$(9.4) \quad h_j^\star(x) := \sum_{\eta \in \mathcal{X}_j} \frac{|h_\eta|}{(1 + 2^j d(x, R_\eta))^\kappa} \leq \sum_{\theta \in \tilde{\mathcal{X}}_j} \frac{|h_\theta|}{(1 + 2^j d(x, R_\theta))^\kappa}$$

and

$$(9.5) \quad \sum_{\eta \in \mathcal{X}_j} |h_\eta| \mathbb{1}_{R_\eta} = \sum_{\theta \in \tilde{\mathcal{X}}_j} |h_\theta| \mathbb{1}_{R_\theta}.$$

Denote  $Y_0 := \{\theta \in \tilde{\mathcal{X}}_j : 2^j \|\theta - x\| \leq \tilde{c}\}$ ,

$$Y_m := \{\theta \in \tilde{\mathcal{X}}_j : \tilde{c}2^{m-1} \leq 2^j \|\theta - x\| \leq \tilde{c}2^m\}, \quad \text{and}$$

$$Q_m := \{y \in \mathbb{R}^d : \|y - x\| \leq \tilde{c}(2^m + 1)2^{-j}\}, \quad m \geq 1.$$

Clearly,  $\#Y_m \leq c2^{md}$ ,  $\cup_{\theta \in Y_m} R_\theta \subset Q_m$ , and  $\tilde{\mathcal{X}} = \cup_{m \geq 0} Y_m$ . Similarly as in (9.3)

$$\begin{aligned} \sum_{\theta \in Y_m} \frac{|h_\theta|}{(1 + 2^j d(x, R_\theta))^\kappa} &\leq c2^{-m\kappa} \sum_{\theta \in Y_m} |h_\theta| \leq c2^{-m\kappa} 2^{md\varrho} \left( \sum_{\theta \in Y_m} |h_\theta|^t \right)^{1/t} \\ &\leq c2^{-m(\kappa-d)} \left( \int_{\cup_{\theta \in Y_m} R_\theta} \sum_{\theta \in Y_m} \mu(R_\theta)^{-1} |h_\theta|^t \mathbb{1}_{R_\theta}(y) w_\alpha(y) dy \right)^{1/t} \\ &\leq c2^{-m(\kappa-d)} \left( \frac{1}{\mu(Q_m)} \int_{Q_m} \left( \sum_{\theta \in Y_m} \left( \frac{\mu(Q_m)}{\mu(R_\theta)} \right)^{1/t} |h_\theta| \mathbb{1}_{R_\theta}(y) \right)^t w_\alpha(y) dy \right)^{1/t}. \end{aligned}$$

Using (4.5) and that  $\cup_{\theta \in Y_m} R_\theta \subset Q_m$  we get

$$\begin{aligned} \frac{\mu(Q_m)}{\mu(R_\theta)} &\leq c \frac{2^{(m-j)d}}{2^{-jd}} \prod_{l=1}^d \left( \frac{x_l + 2^{m-j}}{\theta_l + 2^{-j}} \right)^{2\alpha_j+1} \\ &\leq c 2^{md} \prod_{l=1}^d \left( \frac{\theta_l + 2 \cdot 2^{m-j}}{\theta_l + 2^{-j}} \right)^{2\alpha_j+1} \leq c 2^{m(2|\alpha|+3d)}. \end{aligned}$$

Therefore,

$$\sum_{\theta \in Y_m} \frac{|h_\theta|}{(1 + 2^j d(x, R_\theta))^\kappa} \leq c 2^{-m(\kappa - d - (2|\alpha| + 3d)/t)} \mathcal{M}_t \left( \sum_{\eta \in \mathcal{X}_j} |h_\eta| \mathbb{1}_{R_\eta} \right) (x).$$

Summing up over  $m \geq 0$ , taking into account that  $\kappa > d + (2|\alpha| + 3d)/t$ , and also using (9.4) we arrive at (9.2).  $\square$

**Proof of Lemma 6.11.** For this proof we will need an additional lemma.

**Lemma 9.1.** *Let  $g \in V_{4^j}$ . For any  $\sigma > 0$  and  $L > 0$  we have for  $x', x'' \in 2R_\xi$ , where  $\xi \in \mathcal{X}_j$ ,  $j \geq 0$ ,*

$$(9.6) \quad |g(x') - g(x'')| \leq c 2^j |x' - x''| \sum_{\eta \in \mathcal{X}_j} \frac{|g(\eta)|}{(1 + 2^j \|\xi - \eta\|)^\sigma}$$

and

$$(9.7) \quad |g(x') - g(x'')| \leq c^* 2^{-jL} |x' - x''| \sum_{\eta \in \mathcal{X}_j} \frac{|g(\eta)|}{(1 + 2^j \|\xi - \eta\|)^\sigma}, \quad \text{if } \|\xi\| > (1 + 2\delta)\sqrt{6} \cdot 2^j.$$

Here  $c$  and  $c^*$  depend on  $\alpha$ ,  $d$ ,  $\delta$ , and  $\sigma$  and  $c^*$  depends on  $L$  as well;  $2R_\xi \subset \mathbb{R}^d$  is the set obtained by dilating  $R_\xi$  by a factor of 2 and with the same center.

**Proof.** Let  $\Lambda_{4^j}$  be the kernel from (3.7) with  $n = 4^j$ , where  $\widehat{a}$  is admissible of type (a) with  $v := \delta$ . Then  $\Lambda_{4^j} * g = g$  and  $\Lambda_{4^j}(x, \cdot) \in V_{[(1+\delta)4^j]}$ . Note that  $[(1 + \delta)4^j] + 4^j \leq 2n_j - 1$ . Therefore, by Corollary 5.2

$$g(x) = \int_{\mathbb{R}^d} \Lambda_{4^j}(x, y) g(y) w_\alpha(y) dy = \sum_{\eta \in \mathcal{X}_j} c_\eta \Lambda_{4^j}(x, \eta) g(\eta),$$

where  $c_\eta \sim |R_\eta| W_\alpha(4^j; \eta)$ . From this, we have for  $x', x'' \in 2R_\xi$ ,  $\xi \in \mathcal{X}_j$ ,

$$(9.8) \quad \begin{aligned} |g(x') - g(x'')| &\leq \sum_{\eta \in \mathcal{X}_j} c_\eta |\Lambda_{4^j}(x', \eta) - \Lambda_{4^j}(x'', \eta)| |g(\eta)| \\ &\leq c \|x' - x''\| \sum_{\eta \in \mathcal{X}_j} c_\eta \sup_{x \in 2R_\xi} \|\nabla \Lambda_{4^j}(x, \eta)\| |g(\eta)|. \end{aligned}$$

Note that  $(6(1 + \delta)4^j + 3\alpha + 3)^{1/2} \leq (1 + \delta)\sqrt{6} \cdot 2^j$  for sufficiently large  $j$  (depending on  $\alpha$  and  $\delta$ ). Therefore, using Theorem 3.2 we have for  $\eta \in \mathcal{X}_j$

$$(9.9) \quad \|\nabla \Lambda_{4^j}(x, \eta)\| \leq \frac{c 2^{j(d+1)}}{\sqrt{W_\alpha(4^j; x)} \sqrt{W_\alpha(4^j; \eta)} (1 + 2^j \|x - \eta\|)^\sigma}, \quad x \in \mathbb{R}_+^d,$$

and for any  $L > 0$

$$(9.10) \quad \|\nabla \Lambda_{4^j}(x, \eta)\| \leq \frac{c 2^{-jL}}{(1 + 2^j \|x - \eta\|)^\sigma}, \quad \text{if } \min\{\|x\|, \|\eta\|\} > (1 + \delta)\sqrt{6} \cdot 2^j.$$

Suppose  $\|\xi\| \leq (1+2\delta)\sqrt{6} \cdot 2^j$  and denote  $\mathcal{X}'_j := \{\eta \in \mathcal{X}_j : \|\eta\| \leq (1+\delta)\sqrt{6} \cdot 2^j\}$  and  $\mathcal{X}'' := \mathcal{X}_j \setminus \mathcal{X}'_j$ . We split the sum in (9.8) over  $\mathcal{X}'$  and  $\mathcal{X}''$  to obtain

$$|g(x') - g(x'')| \leq c\|x' - x''\| \left( \sum_{\eta \in \mathcal{X}'_j} \dots + \sum_{\eta \in \mathcal{X}''_j} \dots \right) =: c\|x' - x''\|(\Sigma_1 + \Sigma_2).$$

Using (9.9), (5.17), and that  $c_\eta \sim 2^{-jd}W_\alpha(4^j; \eta)$  for  $\eta \in \mathcal{X}'_j$ , we get

$$(9.11) \quad \begin{aligned} \Sigma_1 &\leq c2^j \sum_{\eta \in \mathcal{X}'_j} \sup_{x \in 2R_\xi} \left( \frac{W_\alpha(4^j; \eta)}{W_\alpha(4^j; x)} \right)^{1/2} \frac{|g(\eta)|}{(1+2^j\|x - \eta\|)^\sigma} \\ &\leq c2^j \sum_{\eta \in \mathcal{X}'_j} \frac{|g(\eta)|}{(1+2^j\|\xi - \eta\|)^{\sigma-2(|\alpha|+d)}} \end{aligned}$$

To estimate  $\Sigma_2$  we use (9.10) and the rough estimate  $c_\eta \leq c2^{jd}$ . We get

$$(9.12) \quad \Sigma_2 \leq c2^{-j(L-d-2\sigma/3)} \sum_{\eta \in \mathcal{X}''_j} \frac{|g(\eta)|}{(1+2^j\|\xi - \eta\|)^\sigma}$$

Here we also used that

$$1+2^j\|\xi - \eta\| \leq 1+2^j(c2^{-j/3} + \|x - \eta\|) \leq c2^{2j/3}(1+2^j\|x - \eta\|) \quad \text{for } x \in 2R_\xi.$$

Estimates (9.11) (with sufficiently large  $\sigma$ ) and (9.12) (with  $L \geq d + 2\sigma/3$ ) imply (9.6).

In the case  $\|\xi\| > (1+2\delta)\sqrt{6} \cdot 2^j$ , we have  $2R_\xi \subset \{x \in \mathbb{R}_+^d : \|x\| \geq (1+\delta)\sqrt{6} \cdot 2^j\}$  for sufficiently large  $j$  and one proceeds just as above but uses only (9.10) as in the estimation of  $\Sigma_2$ . We skip the details.  $\square$

We now proceed with the prove Lemma 6.11. Let  $g \in V_{4^j}$ . Let  $\ell \geq 1$  be sufficiently large (to be determined later on) and denote for  $\xi \in \mathcal{X}_j$

$$(9.13) \quad \mathcal{X}_{j+\ell}(\xi) := \{\eta \in \mathcal{X}_{j+\ell} : R_\eta \cap R_\xi \neq \emptyset\} \quad \text{and}$$

$$(9.14) \quad d_\xi := \sup\{|g(x') - g(x'')| : x', x'' \in R_\eta \text{ for some } \eta \in \mathcal{X}_{j+\ell}(\xi)\}.$$

Our first step is to estimate  $d_\xi$ ,  $\xi \in \mathcal{X}_j$ . Two cases are to be considered here.

*Case I:*  $\|\xi\| \leq (1+3\delta)\sqrt{6} \cdot 2^j$ . By (5.14)

$$(9.15) \quad R_\xi \sim \xi + [-2^{-j}, 2^{-j}]^d \quad \text{and} \quad R_\eta \sim \eta + [-2^{-j-\ell}, 2^{-j-\ell}]^d, \quad \eta \in \mathcal{X}_{j+\ell}(\xi).$$

Hence, for sufficiently large  $\ell$  ( $\ell = \ell(d, \delta)$ ) we have  $\cup_{\eta \in \mathcal{X}_{j+\ell}(\xi)} R_\eta \subset 2R_\xi$ . Now, using estimate (9.6) of Lemma 9.1 with  $\sigma \geq \lambda$  and the fact that  $\text{diam}(R_\eta) \sim 2^{-j-\ell}$  for  $\eta \in \mathcal{X}_{j+\ell}(\xi)$ , we get

$$(9.16) \quad d_\xi \leq c2^{-\ell} \sum_{\eta \in \mathcal{X}_j} \frac{|g(\eta)|}{(1+2^j\|\xi - \eta\|)^\lambda},$$

where  $c > 0$  is a constant independent of  $\ell$ .

*Case II:*  $\|\xi\| > (1+3\delta)\sqrt{6} \cdot 2^j$ . By (5.14) it follows that  $\|x\| > (1+2\delta)\sqrt{6} \cdot 2^j$  for  $x \in \cup_{\eta \in \mathcal{X}_{j+\ell}(\xi)} R_\eta$  if  $j$  is sufficiently large. We apply estimate (9.7) of Lemma 9.1 with  $\sigma \geq \lambda$  and  $L = 1$  to obtain

$$(9.17) \quad d_\xi \leq c2^{-j} \sum_{\eta \in \mathcal{X}_j} \frac{|g(\eta)|}{(1+2^j\|\xi - \eta\|)^\lambda}.$$

We next estimate  $M_\xi^*$ ,  $\xi \in \mathcal{X}_j$  (see (6.14)). Two cases for  $\xi$  occur here.



*Case 1:*  $\|\xi\| \leq (1+4\delta)\sqrt{6} \cdot 2^j$ . Note that (9.15) is again valid. By the definition of  $d_\xi$  in (9.14) it follows that  $M_\xi \leq m_\omega + d_\xi$  for some  $\omega \in \mathcal{X}_{j+\ell}(\xi)$  and hence, using (9.15),

$$M_\xi \leq c \sum_{\omega \in \mathcal{X}_{j+\ell}} \frac{m_\omega}{(1+2^{j+\ell}\|\xi-\omega\|)^\lambda} + d_\xi =: \tilde{m}_\xi + d_\xi, \quad c = c(d, \delta, \lambda, \ell).$$

Consequently,

$$(9.18) \quad M_\xi^* \leq \tilde{m}_\xi^* + d_\xi^*.$$

Denote  $\mathcal{X}'_j := \{\eta \in \mathcal{X}_j : \|\eta\| \leq (1+3\delta)\sqrt{6} \cdot 2^j\}$  and  $\mathcal{X}''_j := \mathcal{X}_j \setminus \mathcal{X}'_j$ . Now, we use (9.16)-(9.17) to obtain

$$\begin{aligned} d_\xi^* &:= \sum_{\eta \in \mathcal{X}_j} \frac{d_\eta}{(1+2^j\|\xi-\eta\|)^\lambda} \leq c2^{-\ell} \sum_{\eta \in \mathcal{X}_j} \sum_{\omega \in \mathcal{X}'_j} \frac{|g(\omega)|}{(1+2^j\|\xi-\eta\|)^\lambda(1+2^j\|\eta-\omega\|)^\lambda} \\ &\quad + c2^{-j} \sum_{\eta \in \mathcal{X}_j} \sum_{\omega \in \mathcal{X}''_j} \frac{|g(\omega)|}{(1+2^j\|\xi-\eta\|)^\lambda(1+2^j\|\eta-\omega\|)^\lambda}. \end{aligned}$$

Replacing  $\mathcal{X}'_j$  and  $\mathcal{X}''_j$  by  $\mathcal{X}_j$  above and shifting the order of summation we get

$$(9.19) \quad \begin{aligned} d_\xi^* &\leq c(2^{-\ell} + 2^{-j}) \sum_{\omega \in \mathcal{X}_j} |g(\omega)| \sum_{\eta \in \mathcal{X}_j} \frac{1}{(1+2^j\|\xi-\eta\|)^\lambda(1+2^j\|\eta-\omega\|)^\lambda} \\ &\leq c(2^{-\ell} + 2^{-j}) \sum_{\omega \in \mathcal{X}_j} \frac{|g(\omega)|}{(1+2^j\|\xi-\omega\|)^\lambda} \leq c(2^{-\ell} + 2^{-j})M_\xi^*. \end{aligned}$$

Here the constant  $c$  is independent of  $\ell$  and  $j$ , and we used that

$$(9.20) \quad \sum_{\eta \in \mathcal{X}_j} \frac{1}{(1+2^j\|\xi-\eta\|)^\lambda(1+2^j\|\eta-\omega\|)^\lambda} \leq \frac{c}{(1+2^j\|\xi-\omega\|)^\lambda} \quad (\lambda > d).$$

This estimate easily follows from the fact that  $\|\xi' - \xi''\| \geq c2^{-j}$  for all  $\xi', \xi'' \in \mathcal{X}_j$ .

To estimate  $\tilde{m}_\xi^*$  we use again (5.14) and (9.20). We get

$$\begin{aligned} \tilde{m}_\xi^* &:= \sum_{\eta \in \mathcal{X}_j} \frac{\tilde{m}_\eta}{(1+2^j\|\xi-\eta\|)^\lambda} \leq c \sum_{\eta \in \mathcal{X}_j} \sum_{\omega \in \mathcal{X}_{j+\ell}} \frac{m_\omega}{(1+2^j\|\xi-\eta\|)^\lambda(1+2^j\|\eta-\omega\|)^\lambda} \\ &\leq c \sum_{\omega \in \mathcal{X}_{j+\ell}} m_\omega \sum_{\eta \in \mathcal{X}_j} \frac{1}{(1+2^j\|\xi-\eta\|)^\lambda(1+2^j\|\eta-\omega\|)^\lambda} \\ &\leq c \sum_{\omega \in \mathcal{X}_{j+\ell}} \frac{m_\omega}{(1+2^j\|\xi-\omega\|)^\lambda} \leq c2^{\ell\lambda} \sum_{\omega \in \mathcal{X}_{j+\ell}} \frac{m_\omega}{(1+2^{j+\ell}\|\theta-\omega\|)^\lambda} = cm_\theta^* \end{aligned}$$

for each  $\theta \in \mathcal{X}_{j+\ell}(\xi)$ . Combining this with (9.18)-(9.19) we obtain

$$M_\xi^* \leq c_1 m_\theta^* + c_2(2^{-\ell} + 2^{-j})M_\xi^* \quad \text{for } \theta \in \mathcal{X}_{j+\ell}(\xi),$$

where  $c_2 > 0$  is independent of  $\ell$  and  $j$ . Choosing  $\ell$  and  $j$  sufficiently large (depending only on  $d, \delta$ , and  $\lambda$ ) this yields  $M_\xi^* \leq cm_\theta^*$  for all  $\theta \in \mathcal{X}_{j+\ell}(\xi)$ . For  $j \leq c$  this relation follows as above but using only (9.6) and taking  $\ell$  large enough. We skip the details. Thus we have shown (6.18) in Case 1.

*Case 2:*  $\|\xi\| > (1 + 4\delta)\sqrt{6} \cdot 2^j$ . Choose  $\ell \geq 1$  the same as in Case 1. Clearly, for sufficiently large  $j$  (depending only on  $d$  and  $\delta$ )  $\|x\| \geq (1 + 3\delta)\sqrt{6} \cdot 2^j$  for  $x \in \cup_{\eta \in \mathcal{X}_{j+\ell}(\xi)} R_\eta$ . Hence, using (9.7) with  $L = 1$ , we have

$$M_\xi \leq m_\omega + c2^{-j} \sum_{\eta \in \mathcal{X}_j} \frac{|g(\eta)|}{(1 + 2^j \|\xi - \eta\|)^\lambda} \leq m_\omega + c2^{-j} M_\xi^* \quad \text{for all } \omega \in \mathcal{X}_{j+\ell}(\xi),$$

where  $c > 0$  is independent of  $j$ . Fix  $\theta \in \mathcal{X}_{j+\ell}(\xi)$  and for each  $\eta \in \mathcal{X}_j$ ,  $\eta \neq \xi$ , choose  $\omega_\eta \in \mathcal{X}_{j+\ell}(\eta)$  so that  $\|\theta - \omega_\eta\| = \min_{\omega \in \mathcal{X}_{j+\ell}(\eta)} \|\theta - \omega\|$ . Then from above

$$(9.21) \quad M_\xi^* \leq \sum_{\eta \in \mathcal{X}_j} \frac{m_{\omega_\eta}}{(1 + 2^j \|\xi - \eta\|)^\lambda} + c2^{-j} \sum_{\eta \in \mathcal{X}_j} \frac{M_\eta^*}{(1 + 2^j \|\xi - \eta\|)^\lambda} =: \Sigma_1 + \Sigma_2.$$

From (2.19) it easily follows that  $\omega_\eta$  from above satisfies  $|\theta - \omega_\eta| \leq c|\xi - \eta|$  and hence

$$(9.22) \quad \Sigma_1 \leq c \sum_{\eta \in \mathcal{X}_j} \frac{m_{\omega_\eta}}{(1 + 2^j \|\theta - \omega_\eta\|)^\lambda} \leq c2^{\ell\lambda} \sum_{\omega \in \mathcal{X}_{j+\ell}} \frac{m_\omega}{(1 + 2^{j+\ell} \|\theta - \omega\|)^\lambda} \leq c_1 m_\theta^*.$$

On the other hand, using Definition 6.9 and (9.20), we have

$$\begin{aligned} \Sigma_2 &\leq c2^{-j} \sum_{\eta \in \mathcal{X}_j} \sum_{\omega \in \mathcal{X}_j} \frac{M_\omega}{(1 + 2^j \|\xi - \eta\|)^\lambda (1 + 2^j \|\eta - \omega\|)^\lambda} \\ &\leq c2^{-j} \sum_{\omega \in \mathcal{X}_j} M_\omega \sum_{\eta \in \mathcal{X}_j} \frac{1}{(1 + 2^j \|\xi - \eta\|)^\lambda (1 + 2^j \|\eta - \omega\|)^\lambda} \\ &\leq c_2 2^{-j} \sum_{\omega \in \mathcal{X}_j} \frac{M_\omega}{(1 + 2^j \|\xi - \omega\|)^\lambda} = c_2 2^{-j} M_\omega^* \end{aligned}$$

with  $c_2 > 0$  independent of  $j$ . Combining this with (9.21)-(9.22) we arrive at

$$M_\xi^* \leq c_1 m_\theta^* + c_2 2^{-j} M_\xi^* \quad \text{for } \theta \in \mathcal{X}_{j+\ell}(\xi).$$

Choosing  $j$  sufficiently large we get  $M_\xi^* \leq c_1 m_\theta^*$  for each  $\theta \in \mathcal{X}_{j+\ell}(\xi)$ . For  $j \leq c$  this estimate follows as in Case 1 but using only (9.6). This completes the proof of Lemma 6.11.  $\square$

**Proof of Lemma 7.5.** Let  $g \in V_{4^j}$  and  $0 < p < \infty$ . We will utilize Definition 6.9 and Lemmas 6.10-6.11. To this end we select  $0 < t < p$  and  $\lambda$  as in Definition 6.9. Set  $M_\xi := \sup_{x \in R_\xi} |g(x)|$ ,  $\xi \in \mathcal{X}_j$ , and  $m_\eta := \inf_{x \in R_\eta} |g(x)|$ ,  $\eta \in \mathcal{X}_{j+\ell}$ , where  $\ell \geq 1$  is the constant from Lemma 6.11. By (1.2) and the properties of the tiles  $R_\xi$  from (5.14)-(5.16) it readily follows that  $W_\alpha(4^{j+\ell}; y) \sim W_\alpha(4^j, \xi)$  for  $y \in R_\xi$ . We now use this, Lemmas 6.10-6.11 and the maximal inequality (4.6) to obtain

$$\begin{aligned} &\left( \sum_{\xi \in \mathcal{X}_j} W_\alpha(4^j; \xi)^{-\rho p/d} \max_{x \in R_\xi} |g(x)|^p \mu(R_\xi) \right)^{1/p} \leq \left\| \sum_{\xi \in \mathcal{X}_j} W_\alpha(4^j; \xi)^{-\rho/d} M_\xi^* \mathbb{1}_{R_\xi} \right\|_p \\ &\leq c \left\| \sum_{\eta \in \mathcal{X}_{j+\ell}} W_\alpha(4^{j+\ell}; \eta)^{-\rho/d} m_\eta^* \mathbb{1}_{R_\eta} \right\|_p \leq c \left\| \mathcal{M}_t \left( \sum_{\eta \in \mathcal{X}_{j+\ell}} W_\alpha(4^{j+\ell}; \eta)^{-\rho/d} m_\eta \mathbb{1}_{R_\eta} \right) \right\|_p \\ &\leq c \left\| \sum_{\eta \in \mathcal{X}_{j+\ell}} W_\alpha(4^{j+\ell}; \eta)^{-\rho/d} m_\eta \mathbb{1}_{R_\eta} \right\|_p \leq c \|W_\alpha(4^j; \cdot)^{-\rho/d} g(\cdot)\|_p. \quad \square \end{aligned}$$

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