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ANISOTROPIC MESHLESS FRAMES ON $\mathbb{R}^n$

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ABSTRACT. We present a construction of anisotropic multiresolution and anisotropic wavelet frames based on multilevel ellipsoid covers (dilations) of $\mathbb{R}^n$. The wavelets we construct are C^∞ functions, can have any prescribed number of vanishing moments and fast decay with respect to the anisotropic quasi-distance induced by the cover. The dual wavelets are also C^∞, with the same number of vanishing moments, but with only mild decay with respect to the quasi-distance. An alternative construction yields a meshless frame whose elements do not have vanishing moments, but do have fast anisotropic decay.

1. INTRODUCTION

Anisotropic phenomena appear in various contexts in mathematical analysis and its applications. The formation of shocks results in jump discontinuities of solutions of hyperbolic conservation laws across lower dimensional manifolds and sharp edges often separate areas of little detail in digital images, to name just two examples. The central objective of this paper is to describe a sufficiently flexible framework for adaptive representations that can efficiently capture anisotropic features of functions e.g. singularities along curves and lower dimensional smooth manifolds.

Anisotropic function spaces on $\mathbb{R}^n$ were extensively studied, beginning with the Russian school in the 1960s (see [30, Chapter 5] for a survey and references therein, specifically Bownik [2] and Bownik and Ho [3]. In Section 2 we review a more general anisotropic framework on $\mathbb{R}^n$ using the multi-level ellipsoid covers introduced in [13] Whereas in previous work the anisotropy is fixed and global over $\mathbb{R}^n$, in our settings only mild conditions are imposed on dilation matrices which are local and allow them to rapidly change from point to point and in depth, from level to level. The ellipsoid covers induce anisotropic quasi-distances on $\mathbb{R}^n$ and together with the usual Lebesgue measure, form spaces of homogeneous type [11, 26].

Once the geometry of an anisotropic space is established, we proceed with the construction of wavelets. The highly anisotropic locally and scale-wise varying structure of the dilations considered here prevents us from using Fourier analysis techniques. Also, the ellipsoid cover which serves as the basis for the construction is of 'meshless' type, i.e. it does not satisfy the exact inclusion property of Euclidian dyadic cubes, where a cube on a higher level is contain in exactly one parent cube in the lower level. However, we can still apply a classical two-step approach to wavelet construction (e.g. [14, 18, 27]), where the first step is to construct a multiresolution analysis and the second step is to create difference operators between each adjacent levels in the multiresolution.

In Section 3, we define a notion of anisotropic multiresolution analysis. We construct operators S_m that are approximation operators associated with a level-of-detail $m \in \mathbb{Z}$,

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which reproduce polynomials up to a specified degree and have arbitrarily high anisotropic regularity, i.e. smoothness and decay with respect to the quasi-distance induced by the cover. In Section 4, with the wavelet operators $D_m := S_{m+1} - S_m$ as starting point, we leverage on the work of Han and Sawyer [22] and the Calderón reproducing formula for spaces of homogeneous type, to construct dual wavelet operators. With these dual operators at hand we show two constructions of anisotropic discrete frames. Let us recall the following definition.

Definition 1.1. A family of elements $\{f_i\}_{i\in I}$ contained in an Hilbert space $\mathcal{H}$ is a *frame* if there exist constants $0 < A \leq B < \infty$ such that such that for any $f \in \mathcal{H}$

$$A\,\|f\|_{\mathcal{H}}^2 \leq \sum_{i\in I} |\langle f, f_i\rangle_{\mathcal{H}}|^2 \leq B\,\|f\|_{\mathcal{H}}^2\,. \tag{1.1}$$

While it is possible to construct an anisotropic orthonormal basis of $L_2(\mathbb{R}^n)$ over an anisotropic triangulation mesh (see [24]), it is still unknown if it is possible to construct a 'meshless' orthonormal basis. Therefore, we focus our attention on frame constructions, in view of the fact that frames can be thought of as some kind of a 'generalized bases', as is evident from (1.1).

First, we apply the tools of [21] and sample from the wavelet kernels D_m discrete wavelet frames that are also smooth and well-localized. The novelty in our setup is that in $\mathbb{R}^n$ there are the notions of approximation order of the operators S_m and the number of vanishing moments of the wavelets. However, we can only prove 'minimal' decay for the dual frame. A second approach we take is to represent the kernels of D_m using 'two level split' elements. We can combine the stability of these elements on each level m with a Littlewood-Paley result for the operators D_m and prove that the 'two level splits', which have fast decay with respect to the anisotropic quasi-distance, are in fact a frame for the entire $L_2(\mathbb{R}^n)$ space.

We note that the geometric setting for our constructions is much more flexible than the setting for the so-called 'irregular frames' (e.g. [1, 7, 8]). Our constructions differ from the Curvelet frame ([4, 5]) in that we describe an adaptive framework while the Curvelet system is 'non-adaptive'. Also, the Curvelet frame contains at each scale and location directional elements at all possible orientations (the number of orientation increases with the scale), while our construction adaptively chooses a 'small' bounded number of elements with a single orientation.

2. Anisotropic ellipsoid covers of $\mathbb{R}^n$

We recall the definitions of [13]. The image of the Euclidian unit ball B^* in $\mathbb{R}^n$ via an affine transform will be called an ellipsoid. For a given ellipsoid θ we let A_θ be an affine transform such that $\theta = A_\theta(B^*)$. Denoting by $v_\theta := A_\theta(0)$ the center of θ we have

$$A_\theta(x) = M_\theta x + v_\theta,$$

where M_θ is a nonsingular $n \times n$ matrix.

Definition 2.1. *We call*

$$\Theta = \bigcup_{m\in\mathbb{Z}} \Theta_m,$$

a discrete multilevel ellipsoid cover of $\mathbb{R}^n$ if the following conditions are obeyed, where $p(\Theta) := \{a_1, \ldots, a_8\}$ are positive constants:

(*a*) *Every level $\Theta_m, m \in \mathbb{Z}$, consists of ellipsoids θ such that*

$$a_1 2^{-m} \leq |\theta| \leq a_2 2^{-m}, \tag{2.1}$$

and Θ_m is a cover of $\mathbb{R}^n$, i.e. $\mathbb{R}^n = \bigcup_{\theta\in\Theta_m}\theta$.

(b) *For each $\theta \in \Theta$ let A_θ be an affine transform associated with θ, of the form*

$$A_\theta(x) = M_\theta x + v_\theta, \quad M_\theta \in \mathbb{R}^{n\times n},$$

such that $\theta = A_\theta(B^*)$ and $v_\theta = A(0)$ is the center of θ. We postulate that for any $\theta \in \Theta_m$ and $\theta' \in \Theta_{m+\nu}$, $\nu \geq 0$, with $\theta\cap\theta' \neq \emptyset$, we have

$$a_3 2^{-a_4\nu} \leq 1\Big/\left\|M_{\theta'}^{-1}M_\theta\right\|_{l_2\to l_2} \leq \left\|M_\theta^{-1}M_{\theta'}\right\|_{l_2\to l_2} \leq a_5 2^{-a_6\nu} \tag{2.2}$$

(c) *Each $\theta \in \Theta_m$ can intersect with at most N_1 ellipsoids from Θ_m.*

(d) *For any $x \in \mathbb{R}^n$ and $m \in \mathbb{Z}$ there exists $\theta \in \Theta_m$ such that $x \in \theta^\diamondsuit$, where $\theta^\diamondsuit$ is the dilated version of θ by a factor of $a_7 < 1$, i.e. $\theta^\diamondsuit = A_\theta(B(0,a_7))$.*

(e) *If $\theta\cap\eta \neq \emptyset$ with $\theta \in \Theta_m$ and $\eta \in \Theta_m \cup \Theta_{m+1}$, then $\theta^\diamondsuit \cap \eta^\diamondsuit \neq \emptyset$, where $\theta^\diamondsuit, \eta^\diamondsuit$ are the dilated versions of θ, η by a factor a_7 as above.*

Remark. As in [13] we can replace condition (2.1) by

$$a_1 2^{-a_0 m} \leq |\theta| \leq a_2 2^{-m a_0},$$

for some $a_0 > 0$. This provides more flexibly in the construction of covers in applications. However, in this work, so as not to burden the reader with more notation, we assume that $a_0 = 1$.

Definition 2.2. We say that

$$\Theta := \bigcup_{t\in\mathbb{R}} \Theta_t$$

is a *continuous multilevel ellipsoid cover* of $\mathbb{R}^n$ if it satisfies the following conditions, where $p(\Theta) := \{a_1, \ldots, a_6\}$ are positive constants:

(a) For every $v \in \mathbb{R}^n$ and $t \in \mathbb{R}$ there exists an ellipsoid $\theta(v,t) \in \Theta_t$ and an affine transform $A_{v,t}(x) = M_{v,t}x + v$ such that $\theta(v,t) = A_{v,t}(B^*)$ and

$$a_1 2^{-t} \leq |\theta(v,t)| \leq a_2 2^{-t}.$$

(b) For any $v, y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $s > 0$, if $\theta(v,t)\cap\theta(y,t+s) \neq \emptyset$, then

$$a_3 2^{-a_4 s} \leq 1/\left\|M_{y,t+s}^{-1}M_{v,t}\right\| \leq \left\|M_{v,t}^{-1}M_{y,t+s}\right\| \leq a_5 2^{-a_6 s}.$$

The discrete and continuous ellipsoid covers induce quasi-distances on $\mathbb{R}^n$. A *quasi-distance* on a set X is a mapping $\rho : X\times X \to [0,\infty)$ that satisfies the following conditions:

(a) $\rho(x,y) = 0 \Leftrightarrow x = y$,

(b) $\rho(x,y) = \rho(y,x)$,

(c) For some $\kappa \geq 1$ and all $x, y \in \mathbb{R}^n$

$$\rho(x,y) \leq \kappa\left(\rho(x,z) + \rho(z,y)\right). \tag{2.3}$$

Let Θ be a cover. We define $\rho : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ by

$$\rho(x,y) = \inf_{\theta\in\Theta}\{|\theta| \,:\, x, y \in \theta\}. \tag{2.4}$$

The following results are proved in [13].

Proposition 2.3. *The function ρ in* (2.4)*, induced by a discrete or a continuous ellipsoid cover, is a quasi-distance on $\mathbb{R}^n$.*

Proposition 2.4. *Any continuous cover can be sampled to a discrete cover, inducing an equivalent quasi-distance.*

Let Θ be an ellipsoid cover inducing a quasi-distance ρ. We denote $B(x,r) := \{y \in \mathbb{R}^n : \rho(x,y) < r\}$. Evidently,

$$B(x,r) = \bigcup_{\theta\in\Theta} \{\theta : \ |\theta| \le r, \ x \in \theta\}.$$

Proposition 2.5. *Let Θ be an ellipsoid cover. For each ball $B(x,r)$, there exist ellipsoids $\theta', \theta'' \in \Theta$, such that $\theta' \subset B(x,r) \subset \theta''$ and $|\theta'| \sim |B(x,r)| \sim |\theta''| \sim r$, where the constants depend on $p(\Theta)$.*

Spaces of homogeneous type were first introduced in [10] (see also [26]) as a means to extend the Calderón-Zygmund theory of singular integral operators to more general settings. Let X be a topological space endowed with a Borel measure μ and a quasi-distance ρ. Assume that the balls $B(x,r) := \{y \in X : \ \rho(x,y) < r\}$, $x \in X$, $r > 0$, form a basis for the topology in X. The space (X, ρ, μ) is said to be of *homogenous type* if there exists a constant A such that for all $x \in X$ and $r > 0$,

$$\mu(B(x,2r)) \le A\mu(B(x,r)). \tag{2.5}$$

If (2.5) holds then μ is said to be a *doubling measure* [29, Chapter 1, 1.1]. A space of homogeneous type is said to be *normal*, if the equivalence $\mu(B(x,r)) \sim r$ holds. Proposition 2.5 gives inequality (2.5) and implies that an ellipsoid cover induces a normal space of homogenous type $(\mathbb{R}^n, \rho, dx)$, where ρ is the quasi-distance (2.4) and dx is the Lebesgue measure.

Let us describe a useful form of covers of $\mathbb{R}^2$. We select all ellipses on levels ≤ 0 to be Euclidian balls. For levels > 0 we allow the ellipses to change from Euclidian balls to ellipses with the 'parabolic scaling' parameters $(a_6, a_4) = (1/3, 2/3)$. This choice of parameters relates to polygonal approximation of a planar curve, with segments of length h and approximation error of $O(h^2)$. Roughly speaking, with this choice we can simulate the performance of polygonal approximation by constructing at the level $m > 0$ roughly $O(2^{m/3})$ 'thin' ellipses of length $\sim 2^{-m/3}$ and width $\sim 2^{-2m/3}$, such they (are aligned with and) cover the function's curve singularities with a 'strip width' of $\sim 2^{-2m/3}$. The actual number of ellipses that are needed depends on the total length of the curve singularities as well as their 'curve smoothness'. Away from the curve singularities, the ellipses can be selected to be Euclidian balls (see also the constructions in [13, Section 7.1]).

We conclude this section by relating the quasi-distances induced by ellipsoid covers with the Euclidian distance. To this end we first require the following definition.

Definition 2.6. Let ρ be a quasi-distance on $\mathbb{R}^n$ and let $\mu = (\mu_0, \mu_1)$, $0 < \mu_0 \le \mu_1 \le 1$. For any $x, y \in \mathbb{R}^n$ and $d > 0$ we define

$$\mu(x,y,d) := \begin{cases} \mu_0 & \rho(x,y) < d, \\ \mu_1 & \rho(x,y) \ge d, \end{cases} \qquad \tilde{\mu}(x,y,d) := \begin{cases} \mu_1 & \rho(x,y) < d, \\ \mu_0 & \rho(x,y) \ge d. \end{cases} \tag{2.6}$$

For $t \in \mathbb{R}$ we define

$$\mu(t) := \begin{cases} \mu_1 & t \le 0, \\ \mu_0 & t > 0, \end{cases} \qquad \tilde{\mu}(t) := \begin{cases} \mu_0 & t \le 0, \\ \mu_1 & t > 0. \end{cases} \tag{2.7}$$

Theorem 2.7. *Let Θ be a discrete ellipsoid cover and ρ the quasi-distance* (2.4). *Denote by $\mu := (\mu_0, \mu_1) = (a_6, a_4)$ where $0 < a_6 \le a_4 \le 1$ are the parameters from Definition 2.1. Then for each fixed $y \in \mathbb{R}^n$ there exist constants $0 < c_1 < c_2 < \infty$ that depend on y and $p(\Theta)$ such that*

$$c_1 \rho(x,y)^{\tilde{\mu}(x,y,1)} \le |x-y| \le c_2 \rho(x,y)^{\mu(x,y,1)}, \quad \forall x \in \mathbb{R}^n, \tag{2.8}$$

where $|x-y|$ is the usual Euclidian distance between x and y.

Proof. Select an ellipsoid $\theta_0 \in \Theta_0$ such that $y \in \theta_0 \in \Theta_0$. For any $x \in \mathbb{R}^n$, let $\theta \in \Theta_m$ such that $\rho(x,y) = \theta$. From condition (2.2) (see also [13, Lemma 2.2]) we obtain

$$|x-y| \le \operatorname{diam}(\theta) \le c \operatorname{diam}(\theta_0) 2^{-\mu(m)m} \le c \operatorname{diam}(\theta_0) a_1^{-\mu(m)} |\theta|^{\mu(m)} \le c_2 \rho(x,y)^{\mu(x,y,1)}.$$

We now prove the right hand side of (2.8). By the minimality of $\theta \in \Theta_m$, there exists $\theta_1 \in \Theta_{m+J}$ such that $y \in \theta_1^{\diamondsuit}$ (the dilated version of θ_1 by a factor a_7) and $x \notin \theta_1$, where $J > 0$ depends only on $p(\Theta)$. Denote by $\sigma_{\min}(\theta_1)$ the minimal semi-axis of θ_1. From (2.2) we get that $\sigma_{\min}(\theta_1) \ge c\sigma_{\min}(\theta_0) 2^{-\tilde{\mu}(m+J)(m+J)}$. Thus,

$$|x-y| \ge (1-a_7)\sigma_{\min}(\theta_1) \ge c2^{-\tilde{\mu}(m+J)(m+J)} \ge c\rho(x,y)^{\tilde{\mu}(m+J)} \ge c_1 \rho(x,y)^{\tilde{\mu}(x,y,1)}.$$

$\diamondsuit$

Remarks.

(1) Observe that in the case where all ellipsoids in Θ_0 are equivalent in shape (for example, to the Euclidian ball), we get that the constants c_1, c_2 in (2.8) depend only on $p(\Theta)$ and not the points y.

(2) In the special case where the ellipsoid cover is composed of Euclidian balls, we have that the parameters in (2.2) satisfy $a_4 = a_6 = 1/n$ and (2.8) is easily verified by

$$|x-y| \sim |\{z: \ |z-x| \le |y-x|\}|^{1/n} \sim \rho(x,y)^{1/n} \sim \rho(x,y)^{\mu(x,y,1)} = \rho(x,y)^{\tilde{\mu}(x,y,1)}.$$

3. Anisotropic multiresolution analysis

We begin with the following generalization to higher orders of the definitions given in [22]. Let $K(x,y)$ be a smooth kernel. For $x,y \in \mathbb{R}^n$, we have the Taylor representation of the kernel about the point x, with y fixed as

$$K(z,y) = T_{r-1,x}(K(\cdot,y))(z) + R_{r,x}(K(\cdot,y))(z), \tag{3.1}$$

where T_{r-1} is the Taylor polynomial of degree $r-1$ (order r) and $R_{r,x}$ is the Taylor remainder of order r.

Definition 3.1. Let $(\mathbb{R}^n, \rho, dx)$ be a normal space of homogeneous type. A sequence of kernel operators $\{S_m\}$, formally defined by $S_m(f)(x) := \int_{\mathbb{R}^n} S_m(x,y) f(y)\,dy$, is a *multiresolution of order* (μ,δ,r), $\mu = (\mu_0,\mu_1)$, $0 < \mu_0 \le \mu_1 \le 1$, $\delta > 0$, $r \in \mathrm{N}$, with respect to ρ, if for some constant $c > 0$ the following conditions are satisfied:

(i) $|S_m(x,y)| \le c\frac{2^{-m\delta}}{(2^{-m}+\rho(x,y))^{1+\delta}}, \quad \forall x,y \in \mathbb{R}^n$.

(ii) For $1 \le k \le r$ and all $x,y,z \in \mathbb{R}^n$,

$$|R_{k,x}(S_m(\cdot,y))(z)| \le c\rho(x,z)^{\mu(x,z,2^{-m})k} \times \left(\frac{2^{-m\delta}}{(2^{-m}+\rho(x,y))^{1+\delta+\mu(x,z,2^{-m})k}} + \frac{2^{-m\delta}}{(2^{-m}+\rho(y,z))^{1+\delta+\mu(x,z,2^{-m})k}} \right),$$

$$|R_{k,y}(S_m(x,\cdot))(z)| \le c\rho(y,z)^{\mu\left(x,z,2^{-m}\right)k} \times \left(\frac{2^{-m\delta}}{(2^{-m}+\rho(x,y))^{1+\delta+\mu(x,z,2^{-m})k}} + \frac{2^{-m\delta}}{(2^{-m}+\rho(x,z))^{1+\delta+\mu(x,z,2^{-m})k}} \right).$$

(iii) For $1 \le k \le r$ and all $x, x', y, y' \in \mathbb{R}^n$

$$
\begin{aligned}
&|R_{k,y}(R_{k,x}(S_m(\cdot,\cdot))(x'))(y')|, |R_{k,x}\left(R_{k,y}\left(S_m\left(\cdot,\cdot\right)\right)\left(y'\right)\right)\left(x'\right)| \\
&\qquad \le c\rho\left(x,x'\right)^{\mu\left(x,x',2^{-m}\right)k} \rho\left(y,y'\right)^{\mu\left(y,y',2^{-m}\right)k} \\
&\qquad \times \left(\frac{2^{-m\delta}}{\left(2^{-m}+\rho\left(x,y\right)\right)^{(1+\delta+\mu(x,x',2^{-m})k+\mu(y,y',2^{-m})k}} \right. \\
&\qquad + \frac{2^{-m\delta}}{\left(2^{-m}+\rho(x,y')\right)^{(1+\delta+\mu(x,x',2^{-m})k+\mu(y,y',2^{-m})k}} \\
&\qquad + \frac{2^{-m\delta}}{\left(2^{-m}+\rho(x',y)\right)^{(1+\delta+\mu(x,x',2^{-m})k+\mu(y,y',2^{-m})k}} \\
&\qquad \left. + \frac{2^{-m\delta}}{\left(2^{-m}+\rho(x',y')\right)^{(1+\delta+\mu(x,x',2^{-m})k+\mu(y,y',2^{-m})k}} \right)
\end{aligned}
$$

[To clarify our notation, denote $g_m\left(x,x',y\right) := R_{k,x}\left(S_m\left(\cdot,y\right)\right)\left(x'\right)$, then for fixed $x, x' \in \mathbb{R}^n$, $R_{k,y}\left(R_{k,x}\left(S_m\left(\cdot,\cdot\right)\right)\left(x'\right)\right)\left(y'\right) = R_{k,y}\left(g_m\left(x,x',\cdot\right)\right)\left(y'\right)$].

(iv) $P\left(x\right) = \int_{\mathbb{R}^n} S_m\left(x,y\right) P\left(y\right) dy$ and $P\left(y\right) = \int_{\mathbb{R}^n} S_m\left(x,y\right) P\left(x\right) dx$, for every polynomial $P \in \Pi_{r-1}$, where Π_{r-1} are the polynomials of total degree $r-1$.

Remarks.

(1) We shall use the fact that condition (ii) implies

$$
\begin{aligned}
&|R_{r,x}\left(S_m\left(\cdot,y\right)\right)\left(z\right)| \le c\rho\left(x,z\right)^{\mu_0 k} \frac{2^{-m\delta}}{\left(2^{-m}+\rho(x,y)\right)^{1+\delta+\mu_0 k}}, \\
&\qquad \text{if } \rho\left(x,z\right) \le \tfrac{1}{2\kappa}\left(2^{-m}+\rho\left(x,y\right)\right), \\
&|R_{r,y}\left(S_m\left(x,\cdot\right)\right)\left(z\right)| \le c\rho\left(y,z\right)^{\mu_0 k} \frac{2^{-m\delta}}{\left(2^{-m}+\rho(x,y)\right)^{1+\delta+\mu_0 k}}, \\
&\qquad \text{if } \rho\left(y,z\right) \le \tfrac{1}{2\kappa}\left(2^{-m}+\rho\left(x,y\right)\right),
\end{aligned}
\tag{3.2}
$$

and condition (iii) implies

$$
|R_{k,y}\left(R_{k,x}\left(S_m\left(\cdot,\cdot\right)\right)\left(x'\right)\right)\left(y'\right)| \le c\rho\left(x,x'\right)^{\mu_0 k} \rho\left(y,y'\right)^{\mu_0 k} \frac{2^{-m\delta}}{\left(2^{-m}+\rho\left(x,y\right)\right)^{1+\delta+2\mu_0 k}},
\tag{3.3}
$$

if $\rho\left(x,x'\right) \le \frac{1}{2\kappa}\left(2^{-m}+\rho\left(x,y\right)\right)$ and $\rho\left(y,y'\right) \le \frac{1}{2\kappa}\left(2^{-m}+\rho\left(x,y\right)\right)$, where κ is given in (2.3).

(2) The definition given in [22] corresponds to the case $0 < \delta < r = 1$. There, (3.2) and (3.3) are used in place of conditions (ii) and (iii) here.

(3) See [15, Lemma 2.2] or [22] for Coifman's construction of a multiresolution analysis of order $r = 1$ for arbitrary spaces of homogeneous type.

Let Θ be a discrete ellipsoid cover (see Definition 2.1). Our goal is to construct a multiresolution $\{S_m\}_{m\in\mathbb{Z}}$ that satisfies the above properties for arbitrary $r \ge 1$, where the quasi-distance ρ is induced by the cover . We shall first construct for each level $m \in \mathbb{Z}$ a stable basis Φ_m whose elements are C^∞ 'bumps' that reproduce polynomials and are supported on the ellipsoids of Θ_m (the construction is a modification of the one given in [13]. To this end, we split Θ_m into no more than N_1 disjoint sets $\{\Theta_m^\nu\}_{\nu=1}^{N_1}$ (N_1 appears in condition (c) in Definition 2.1), so that neither two ellipsoids $\theta', \theta'' \in \Theta_m$, with $\theta' \cap \theta'' \ne \emptyset$ are of the same color.

Remark 3.2. In Section 4.4, where we require the stability of the 'two-level splits' of [13], we shall need a stronger coloring scheme, where two intersecting ellipsoids from adjacent levels also have different colors.

The method we employ here to ensure stability of Φ_m is to construct the 'core' part of each basis function supported on an ellipsoid as a rational function, whose nominator is a polynomial of a certain degree which is different from the degrees of the nominators of its neighbors, i.e. the basis functions supported on neighbor ellipsoids. This construction will give local linear independence of neighbor basis function and eventually lead to the global stability of Φ_m. To this end we first recall the well known *up-function* can be defined by

$$\mathrm{up} = \frac{\mathbb{1}_{[-1/2,1/2]}}{2^0} * \frac{\mathbb{1}_{[-1/2^2,1/2^2]}}{2^1} * \frac{\mathbb{1}_{[-1/2^3,1/2^3]}}{2^2} * \cdots \tag{3.4}$$

It is easy to see that for any univariate polynomial P and segment $[a,b]$, the function $P*\mathbb{1}_{[a,b]}$ is also a polynomial and $\deg\left(P*\mathbb{1}_{[a,b]}\right) = \deg P$. Hence, $P*up$ is also a polynomial and $\deg\left(P*up\right) = \deg P$.

Choose $l>1$ so that $2^{-l} \le (1-a_7)/4$, where $a_7 < 1$ is from condition (d) in Definition 2.1. From the Fourier representation of (3.4) it readily follows that

$$\mathrm{up}\left(2^l\cdot\right) = \frac{\mathbb{1}_{[-1/2^{l+1},1/2^{l+1}]}}{2^l} * \frac{\mathbb{1}_{[-1/2^{l+2},1/2^{l+2}]}}{2^{l+1}} * \frac{\mathbb{1}_{[-1/2^{l+3},1/2^{l+3}]}}{2^{l+2}} * \cdots$$

Denote

$$h_\nu(t) := \left(\left(1-2^{-l}\right)^2 - t^2\right)_+^{\nu r}, \quad t\in\mathbb{R}, \quad 1\le\nu\le N_1,$$

and consider the function $H_\nu := h_\nu * \mathrm{up}\left(2^l\cdot\right)$. Clearly, $H_\nu \in C^\infty$, $\operatorname{supp} H_\nu = [-1,1]$, H_ν is even and the restriction of H_ν on $\left[-1+2^{-l+1}, 1-2^{-l+1}\right]$ is a polynomial of degree precisely $2\nu r$. We define

$$\phi_\nu(x) := H_\nu(|x|), \quad x\in\mathbb{R}^n. \tag{3.5}$$

From above it follows that $\phi_\nu \in C^\infty(\mathbb{R}^n)$, $\phi_\nu \ge 0$, $\operatorname{supp}\phi_\nu = \overline{B^*}$ with B^* being the Euclidean unit ball in $\mathbb{R}^n$ and the restriction of ϕ_ν is a polynomial of degree $2\nu r$ on $B(0,(a_7+1)/2)$. In addition,

$$\phi_\nu|_{B(0,a_7)} \ge c_1 > 0, \quad c_1 = c_1(N_1, r). \tag{3.6}$$

For any ellipsoid θ let A_θ be the affine transform satisfying $A_\theta(B^*) = \theta$ and let $\phi_\theta := \phi_\nu \circ A_\theta^{-1}$, if $\theta\in\Theta_m^\nu$. It is standard to form a partition of unity $\{\tilde\phi_\theta\}_{\theta\in\Theta_m}$ by setting

$$\tilde\phi_\theta := \frac{\phi_\theta}{\sum\limits_{\theta'\in\Theta_m}\phi_{\theta'}}. \tag{3.7}$$

Observe that property (d) of ellipsoid covers (see Definition 2.1) together with (3.6) ensure that $0 < c' \le \sum_{\theta\in\Theta_m}\phi_\theta(x) \le c''$, for all $x\in\mathbb{R}^n$ and hence $\tilde\phi_\theta$ is well defined and $\sum_{\theta\in\Theta_m}\tilde\phi_\theta = 1$.

Fix $1\le\nu\le N_1$. Suppose $\{P_\beta : \beta\in\mathrm{N}^n, |\beta| = \beta_1+\cdots+\beta_n \le r-1\}$ is an orthonormal basis for Π_{r-1} in the weighted norm $\|f\|_{L_2(B^*,\phi_\nu)} := \|f\phi_\nu\|_{L_2(B^*)}$. Then for any $\theta\in\Theta_m^\nu$ and $\beta\in\mathrm{N}^n$, $|\beta|<r$, we define

$$P_{\theta,\beta} := |\theta|^{-1/2} P_\beta \circ A_\theta^{-1}, \tag{3.8}$$

and set

$$\varphi_{\theta,\beta} := P_{\theta,\beta}\tilde\phi_\theta. \tag{3.9}$$

To simplify our notation, we denote

$$\Lambda_m := \{\lambda := (\theta, \beta) : \theta \in \Theta_m, |\beta| < r\}, \tag{3.10}$$

and if $\lambda = (\theta, \beta)$ we shall denote by θ_λ and β_λ the components of λ.

Notice that from our construction $\|\varphi_\lambda\|_2 = 1$ and in general $\|\varphi_\lambda\|_p \sim |\theta_\lambda|^{1/p-1/2}$, $0 \le p \le \infty$. In going further we define the mth level basis Φ_m by

$$\Phi_m := \{\varphi_\lambda : \lambda \in \Lambda_m\} \text{ and set } \mathcal{S}_m^r := \overline{\operatorname{span}}(\Phi_m). \tag{3.11}$$

It is easy to see that $\Pi_{r-1} \subset \mathcal{S}_m^r$, since for any polynomial $P \in \Pi_{r-1}$ and $\theta \in \Theta_m$ there exist a representation $P = \sum_{|\beta|<r} c_{\theta,\beta} P_{\theta,\beta}$ and therefore

$$P = \sum_{\theta \in \Theta_m} P \tilde{\phi}_\theta = \sum_{\theta \in \Theta_m, |\beta|<r} c_{\theta,\beta} P_{\theta,\beta} \tilde{\phi}_\theta = \sum_{\theta \in \Theta_m, |\beta|<r} c_{\theta,\beta} \varphi_{\theta,\beta} = \sum_{\lambda \in \Lambda_m} c_\lambda \varphi_\lambda. \tag{3.12}$$

As we already discussed, the stability of Φ_m is critical for our further development.

Proposition 3.3. *If $f \in \mathcal{S}_m^r \cap L_p$, $0 < p \le \infty$, and $f = \sum_{\lambda \in \Lambda_m} c_\lambda \varphi_\lambda$, then*

$$\|f\|_p \sim \Big(\sum_{\lambda \in \Lambda_m} \|c_\lambda \varphi_\lambda\|_p^p \Big)^{1/p} \sim 2^{m\left(\frac{1}{2}-\frac{1}{p}\right)} \Big(\sum_{\lambda \in \Lambda_m} |c_\lambda|^p \Big)^{1/p}, \tag{3.13}$$

with the obvious modification when $p = \infty$ and where the constant of equivalency depend only on $p(\Theta), n, r, p$ and our choice of 'bumps' $\{\phi_\nu\}_{\nu=1,\dots,N_1}$.

The proof of the proposition is simply a repetition of the proof of Theorem 3.2 in [13]. It relies on the fact that as in [13], each ϕ_θ is a polynomial on the dilated version of θ by a factor of $(a_7 + 1)/2$. We omit the proof.

To construct well localized kernels $S_m(x, y)$ which reproduce polynomials we need to construct an appropriate dual basis to Φ_m. Let G_m be the Gram matrix given by

$$G_m := [A_{\lambda,\lambda'}]_{\lambda,\lambda' \in \Lambda_m}, \quad A_{\lambda,\lambda'} := \langle \varphi_\lambda, \varphi_{\lambda'} \rangle := \int_{\mathbb{R}^n} \varphi_\lambda \varphi_{\lambda'}.$$

By Proposition 3.3, for any sequence $\alpha = (\alpha_\lambda)_{\lambda \in \Lambda_m}$ in $l_2(\Lambda_m)$ we have

$$c_1 \|\alpha\|_{l_2} \le \langle G_m \alpha, \alpha \rangle = \Big\| \sum_{\lambda \in \Lambda_m} \alpha_\lambda \varphi_\lambda \Big\|_2 \le c_2 \|\alpha\|_{l_2},$$

where the constants $c_1, c_2 > 0$ do not depend on α or m. Thus the operator $G_m : l_2 \to l_2$ with matrix G_m is symmetric, positive and $c_1 I \le G_m \le c_2 I$. Therefore, G_m^{-1} exists and $c_2^{-1} I \le G_m \le c_1^{-1} I$. Denote by $G_m^{-1} := [B_{\lambda,\lambda'}]_{\lambda,\lambda' \in \Lambda_m}$ the matrix of the operator G_m^{-1}.

We now introduce a *graph-distance* $\tilde{d}_m(\cdot,\cdot)$ on Λ_m. To this end we first define the graph-distance $d_m(\theta, \theta')$ between any $\theta, \theta' \in \Theta_m$ as the length of the shortest *chain* connecting θ and θ'. A chain is a list of ellipsoids in Θ_m where each consecutive ellipsoids have a non-empty intersection and its length is the number of elements $-$ 1. Evidently, d_m is a distance on Θ_m. Let us order in a sequence, indexed by $0, 1, \dots$, the multi-indices $\beta \in \mathbb{N}^n$ in such a way that if $N(\beta)$ denotes the index of β then $N(\beta) < N(\beta')$ for $|\beta| < |\beta'|$. Denote also $N_{\max} := \max_{|\beta|<r} N(\beta) + 1$. After this preparation, we define the graph distance $\tilde{d}_m(\lambda, \lambda')$ between any $\lambda, \lambda' \in \Lambda_m$ by

$$\tilde{d}_m(\lambda, \lambda') := N_{\max} d_m(\theta_\lambda, \theta_{\lambda'}) + |N(\beta_\lambda) - N(\beta_{\lambda'})|.$$

It is readily seen that $\tilde{d}_m(\cdot,\cdot)$ is a true distance on Λ_m, which is dominated by the graph distance between the ellipsoids. Applying a generalization, given in [25], of a well-known result of Demko on the inverses of band matrices, we arrive at the following lemma (see also [25, Lemma 3.6].

Theorem 3.4. *There exist constants $0 < q < 1$ and $c > 0$ depending only on $p(\Theta)$, r and our choice of $\{\phi_\nu\}_{\nu=1,\dots,N_1}$, such that the following estimates hold for the entries of G_m^{-1}, $m \in \mathbb{Z}$*

$$|B_{\lambda,\lambda'}| \le cq^{\tilde{d}_m(\lambda,\lambda')} \le cq^{d_m(\theta_\lambda,\theta_{\lambda'})}, \quad \lambda,\lambda' \in \Lambda_m. \tag{3.14}$$

In going further, we need an estimate of the entries $B_{\lambda,\lambda'}$ using the quasi-distance. First we need the following result given in [13, Lemma 2.8].

Lemma 3.5. *There is an integer $J > 0$ depending only on $p(\Theta)$ such that for any two ellipsoids $\theta \in \Theta_m$ and $\theta' \in \Theta_{m+\nu}$, $\nu > 0$, such that $\theta \cap \theta' \neq \emptyset$, there exists an ellipsoid $\eta \in \Theta_{m-J}$ such that $\theta, \theta' \subset \eta$.*

Lemma 3.6. *There exist constants $0 < q_*, \gamma < 1$ and $c > 0$ depending only on $p(\Theta)$ and r such that for any entry $B_{\lambda,\lambda'}$, $\lambda, \lambda' \in \Lambda_m$ and points $x \in \theta_\lambda$, $y \in \theta_{\lambda'}$*

$$|B_{\lambda,\lambda'}| \le cq_*^{(2^m\rho(x,y))^\gamma}. \tag{3.15}$$

Proof. Let $\lambda, \lambda' \in \Lambda_m$. There exists a connected chain of ellipsoids in Θ_m of length $d_m(\theta_\lambda, \theta_{\lambda'})$ that starts at θ_λ and ends in $\theta_{\lambda'}$. By Lemma 3.5, we can find a connected chain of ellipsoids in Θ_{m-J} of length $\lceil d_m(\theta_\lambda,\theta_{\lambda'})/2\rceil$ whose first element contains θ_λ and the last $\theta_{\lambda'}$. After at most $L := 2\log_2(d_m(\theta_\lambda,\theta_{\lambda'}))$ such iterations, we obtain an ellipsoid $\eta \in \Theta_{m-LJ}$ such that $\theta_\lambda, \theta_{\lambda'} \subset \eta$ and therefore

$$\rho(x,y) \le |\eta| \le a_2 2^{-(m-LJ)} = a_2 2^{-m} 2^{\log_2(d_m(\theta_\lambda,\theta_{\lambda'}))2J} = a_2 2^{-m} d_m(\theta_\lambda,\theta_{\lambda'})^{2J}. \tag{3.16}$$

Denoting $q_* := q^{a_2^{-1/2J}}$, where q is defined in (3.14) and $\gamma := 1/2J$, we conclude that (3.15) holds by combining (3.14) and (3.16)

$$|B_{\lambda,\lambda'}| \le cq^{d_m(\theta_\lambda,\theta_{\lambda'})} \le cq^{\left(a_2^{-1}2^m\rho(x,y)\right)^{1/2J}} = cq_*^{(2^m\rho(x,y))^\gamma}.$$

$\diamondsuit$

Definition 3.7. We define the *dual basis* $\tilde{\Phi}_m := \{\tilde{\varphi}_\lambda\}_{\lambda\in\Lambda_m}$ by

$$\tilde{\varphi}_\lambda := \sum_{\lambda'\in\Lambda_m} B_{\lambda,\lambda'}\varphi_{\lambda'} \quad \lambda \in \Lambda_m. \tag{3.17}$$

For $\lambda \in \Lambda_m$, let x_0 be any point in θ_λ. Combining (3.15) and (3.17) we see that

$$|\tilde{\varphi}_\lambda(x)| \le c2^{-m/2} \sum_{x\in\theta_{\lambda'}} |B_{\lambda,\lambda'}| \le c2^{-m/2} q_*^{(2^m\rho(x,x_0))^\gamma}. \tag{3.18}$$

Therefore, each $\tilde{\varphi}_\lambda$ has fast decay with respect to the quasi-distance induced by Θ and so by (2.8) it also has fast decay with respect to the Euclidian distance (in fact it is in the Schwartz class S, we omit the proof). Also,

$$\langle \varphi_\lambda, \tilde{\varphi}_{\lambda'}\rangle = \sum_{\lambda''\in\Lambda_m} B_{\lambda',\lambda''}\langle \varphi_\lambda, \varphi_{\lambda''}\rangle = \left(G_m^{-1}G_m\right)_{\lambda',\lambda} = \delta_{\lambda,\lambda'}.$$

We use the bases Φ_m and $\tilde{\Phi}_m$ to define the multiresolution kernel operators $\{S_m\}_{m\in\mathbb{Z}}$ by

$$S_m(x,y) = \sum_{\lambda\in\Lambda_m} \varphi_\lambda(x)\tilde{\varphi}_\lambda(y). \tag{3.19}$$

Our next step is to show that $\{S_m\}_{m\in\mathbb{Z}}$ form a high order multiresolution analysis (see Definition 3.1). As we shall see the parameter μ depends on the parameters of the cover. We begin with the following result (see [24, Lemma 4.2] for the case $r=2$ and triangulation meshes).

Lemma 3.8. *Let Θ be an ellipsoid cover of $\mathbb{R}^n$, denote $\mu := (a_6, a_4)$ (see Definition 2.1) and let $k \in \mathbb{N}$. For any $\lambda \in \Lambda_m$ and $x, z \in \mathbb{R}^n$*

$$|R_{k,x}(\varphi_\lambda, z)| \le c 2^{m/2} \left(2^m \rho(x,z)\right)^{\mu\left(x,z,2^{-m}\right)k} \tag{3.20}$$

where $R_{k,x}(f,z)$ is the Taylor remainder of order k about the point x and at the point z. The constant depends on the parameters of the cover, k, r, n and the choice of $\{\phi_\nu\}_{\nu=1,\dots,N_1}$.

Proof. Assume first that $\theta := \theta_\lambda \in \Theta_0$ and that $\theta = B^*$, where B^* is the Euclidian unit ball in $\mathbb{R}^n$. Denote $|f|_{W_k^\infty} := \sum_{|\gamma|=k} \|\partial^\gamma f\|_\infty$. Evidently, in this special case, $|\varphi_\lambda|_{W_k^\infty} \le c^*$ with c^* depending on the aforementioned parameters. By definition there exists an ellipsoid $\tilde\theta \in \Theta_j$, for some $j \in \mathbb{Z}$, such that $\rho(x,z) = |\tilde\theta|$. Since we may assume that either x or z are in θ (otherwise $R_{k,x}(\varphi_\lambda, z) = 0$ and (3.20) is obvious) we get that $\theta \cap \tilde\theta \ne \emptyset$. We may consider two cases:

Case 1: $j \ge 0$. Since $\tilde\theta \cap B^* \ne \emptyset$ then by condition (2.2) (see [13, Lemma 2.2]) we have $|x-z| \le \operatorname{diam}(\tilde\theta) \le c 2^{-a_6 j}$. Also, since $\tilde\theta \in \Theta_j$, we have by (2.1) that $|\tilde\theta| \ge a_1 2^{-j}$. Combining these last two estimates yields

$$|R_{k,x}(\varphi_{B^*,\beta}, z)| \le c\, |\varphi_{B^*,\beta}|_{W_k^\infty} |x-z|^k \le c 2^{-a_6 jk} \le c |\tilde\theta|^{a_6 k} \le c \rho(x,z)^{a_6 k}.$$

Case 2: $j < 0$. Since $\tilde\theta \cap B^* \ne \emptyset$ then by condition (2.2) we have $|x-z| \le \operatorname{diam}(\tilde\theta) \le C 2^{-a_4 j}$. Similarly as above one arrives at

$$|R_{k,x}(\varphi_{B^*,\beta}, z)| \le c \rho(x,z)^{a_4 k}.$$

These last two estimates prove (3.20) for the case $\theta_\lambda \in \Theta_0$ and $\theta_\lambda = B^*$. We now consider the case where both the ellipsoid and the cover are arbitrary. We first observe that for any $f \in W_k^\infty$ and affine transform A, one has for $x, z \in \mathrm{R}^n$

$$T_{k-1,x}(f \circ A, z) = T_{k-1,A(x)}(f, A(z)) \text{ and } R_{k,x}(f \circ A, z) = R_{k,A(x)}(f, A(z)). \tag{3.21}$$

As in Definition 2.1, let A_θ be an affine transform such that $\theta = A_\theta(B^*)$. Evidently, $\Theta^* := \left\{A^{-1}(\eta)\right\}_{\eta\in\Theta}$ is an ellipsoid cover of $\mathbb{R}^n$ with the same parameters as Θ. Denote by $\rho^*(\cdot,\cdot)$ the quasi-distance induced by Θ^*. It is easy to see that

$$\rho^*\left(A^{-1}(x), A^{-1}(z)\right) = |\theta|^{-1} \rho(x,z). \tag{3.22}$$

Denote $\varphi_{B^*,\beta} := \tilde\phi_{B^*} P_\beta$ and notice that $\varphi_{\theta,\beta} = |\theta|^{-1/2} \varphi_{B^*,\beta} \circ A_\theta^{-1}$. Observing that (3.20) holds for the special case of $A^{-1}(\theta) = B^* \in \Theta^*$, we use (3.22) and (3.21) to obtain

$$\begin{aligned}
|R_{k,x}(\varphi_{\theta,\beta}, z)| &= |\theta|^{-1/2} \left|R_{k,A_\theta^{-1}(x)}\left(\varphi_{B^*,\beta}, A_\theta^{-1}(z)\right)\right| \\
&\le c\, |\theta|^{-1/2} \rho^*\left(A_\theta^{-1}(x), A_\theta^{-1}(z)\right)^{\mu\left(A_\theta^{-1}(x), A_\theta^{-1}(z), 1\right)} \\
&= c\, |\theta|^{-1/2} \left(|\theta|^{-1} \rho(x,z)\right)^{\mu\left(x,z,2^{-m}\right)}.
\end{aligned}$$

The proof of Lemma 3.8 is complete.

$\diamondsuit$

Theorem 3.9. *Suppose Θ is a discrete ellipsoid cover of $\mathbb{R}^n$, denote $\mu := (a_6, a_4)$ and let S_m, $m \in \mathbb{Z}$, be defined as in (3.19). Then there exist $0 < q_*, \gamma < 1$ and $c > 0$ such that for any $x, x', y, y', z \in \mathbb{R}^n$*

$$|S_m(x,y)| \le c2^m q_*^{(2^m\rho(x,y))^\gamma}, \tag{3.23}$$

$$|R_{k,x}(S_m(\cdot,y),z)| \le c2^m (2^m\rho(x,z))^{\mu(x,z,2^{-m})k} \left(q_*^{(2^m\rho(x,y))^\gamma} + q_*^{(2^m\rho(y,z))^\gamma}\right), \tag{3.24}$$

$$|R_{k,y}(S_m(x,\cdot),z)| \le c2^m (2^m\rho(y,z))^{\mu(x,z,2^{-m})k} \left(q_*^{(2^m\rho(x,y))^\gamma} + q_*^{(2^m\rho(x,z))^\gamma}\right), \tag{3.25}$$

$$\begin{aligned} &|R_{k,y}R_{k,x}(S_m(\cdot,\cdot),z)| = |R_{k,x}R_{k,y}(S_m(\cdot,\cdot),z)| \\ &\le c2^m (2^m\rho(x,x'))^{\mu(x,x',2^{-m})k} (2^m\rho(y,y'))^{\mu(y,y',2^{-m})k} \\ &\times \left(q_*^{(2^m\rho(x,y))^\gamma} + q_*^{(2^m\rho(x,y'))^\gamma} + q_*^{(2^m\rho(x',y))^\gamma} + q_*^{(2^m\rho(x',y'))^\gamma}\right). \end{aligned} \tag{3.26}$$

Proof. By (3.17) and (3.19) the kernel $S_m(x,y)$ has a representation

$$S_m(x,y) = \sum_{\lambda,\lambda'\in\Lambda_m} B_{\lambda,\lambda'}\varphi_\lambda(x)\varphi_{\lambda'}(y). \tag{3.27}$$

Applying (3.15) we obtain (3.23)

$$|S_m(x,y)| \le \sum_{x\in\theta_\lambda, y\in\theta_{\lambda'}} |B_{\lambda,\lambda'}|\,|\varphi_\lambda(x)|\,|\varphi_{\lambda'}(y)| \le c2^m q_*^{(2^m\rho(x,y))^\gamma}.$$

For the proof of (3.24), we apply (3.27), (3.20) and (3.15)

$$\begin{aligned} |R_{k,x}(S_m(\cdot,y))(z)| &\le \sum_{x\in\theta_\lambda \text{ or } z\in\theta_\lambda} \sum_{y\in\theta_{\lambda'}} |B_{\lambda,\lambda'}|\,|R_{r,x}(\varphi_\lambda,z)|\,|\varphi_{\lambda'}(y)| \\ &\le c2^m (2^m\rho(x,z))^{\mu(x,z,2^{-m})} \left(\sum_{x\in\theta_\lambda, y\in\theta_{\lambda'}} |B_{\lambda,\lambda'}| + \sum_{z\in\theta_\lambda, y\in\theta_{\lambda'}} |B_{\lambda,\lambda'}|\right) \\ &\le c2^m (2^m\rho(x,z))^{\mu(x,z,2^{-m})} \left(q_*^{(2^m\rho(x,y))^\gamma} + q_*^{(2^m\rho(y,z))^\gamma}\right). \end{aligned}$$

The proof of (3.25) is similar. Finally, we prove (3.26) using the same technique

$$\begin{aligned} &|R_{r,y}(R_{r,x}(S_m(\cdot,\cdot))(x'))(y')| \\ &\le \sum_{x\in\theta_\lambda \text{ or } x'\in\theta_\lambda} \sum_{y\in\theta_{\lambda'} \text{ or } y'\in\theta_{\lambda'}} |B_{\lambda,\lambda'}|\,|R_{r,x}(\varphi_\lambda,x')|\,|R_{r,y}(\varphi_{\lambda'},y')| \\ &\le c2^m (2^m\rho(x,x'))^{\mu(x,x',2^{-m})} (2^m\rho(y,y'))^{\mu(y,y',2^{-m})} \\ &\times \left(\sum_{x\in\theta_\lambda, y\in\theta_{\lambda'}} |B_{\lambda,\lambda'}| + \sum_{x\in\theta_\lambda, y'\in\theta_{\lambda'}} |B_{\lambda,\lambda'}| + \sum_{x'\in\theta_\lambda, y\in\theta_{\lambda'}} |B_{\lambda,\lambda'}| + \sum_{x'\in\theta_\lambda, y'\in\theta_{\lambda'}} |B_{\lambda,\lambda'}|\right) \\ &\le c2^m (2^m\rho(x,x'))^{\mu(x,x',2^{-m})} (2^m\rho(y,y'))^{\mu(y,y',2^{-m})} \\ &\quad\times \left(q_*^{(2^m\rho(x,y))^\gamma} + q_*^{(2^m\rho(x,y'))^\gamma} + q_*^{(2^m\rho(x',y))^\gamma} + q_*^{(2^m\rho(x',y'))^\gamma}\right). \end{aligned}$$

$\diamondsuit$

We can now prove that our construction is indeed a high order multiresolution.

Corollary 3.10. *For a discrete ellipsoid cover Θ, the kernels (3.19) are a multiresolution with respect to the quasi-distance (2.4) induced by the cover. The vector μ can be taken as $\mu = (a_6, a_4)$, the parameter δ can be any positive and the parameter r is the total order of the polynomials (3.8) used in the construction of the local ellipsoid 'bumps'.*

Proof. The corollary is immediate from the previous theorem using standard techniques, but we give the proof for the sake of completeness. For any $\tilde{\delta} > 0$ denote $\tilde{q} := q_*^{1/\tilde{\delta}}$, where q_* is given by (3.15). Evidently, for any $0 < \tilde{q}, \gamma < 1$ there exists a constant $c_1(\tilde{q}, \gamma) > 0$ such that $\tilde{q}^{t^\gamma} \le c_1 (1+t)^{-1}$, $\forall t \ge 0$. Therefore, for all $m \in \mathbb{Z}$, $x, y \in \mathbb{R}^n$ we have

$$q_*^{(2^m \rho(x,y))^\gamma} = \tilde{q}^{(2^m \rho(x,y))^\gamma \tilde{\delta}} \le c_1^{\tilde{\delta}} \left(\frac{1}{1 + 2^m \rho(x,y)} \right)^{\tilde{\delta}} = c \frac{2^{-m\tilde{\delta}}}{(2^{-m} + \rho(x,y))^{\tilde{\delta}}}. \tag{3.28}$$

Thus, for any $\delta > 0$, setting $\tilde{\delta} = 1 + \delta$ in (3.28), we get from (3.23)

$$|S_m(x,y)| \le c 2^m q_*^{(2^m \rho(x,y))^\gamma} \le c 2^m \tfrac{2^{-m(1+\delta)}}{(2^{-m} + \rho(x,y))^{1+\delta}} = c \tfrac{2^{-m\delta}}{(2^{-m} + \rho(x,y))^{1+\delta}},$$

which is property (i) in Definition 3.1. Property (ii) is proved similarly, by applying (3.24) or (3.25) for $1 \le k \le r$ and setting $\tilde{\delta} = 1 + \delta + \mu_1 k$, i.e,

$$\begin{aligned}
|R_{k,x}(S_m(\cdot, y), z)| &\le c 2^m (2^m \rho(x,z))^{\mu(x,z,2^{-m})k} \left(q_*^{(2^m \rho(x,y))^\gamma} + q_*^{(2^m \rho(y,z))^\gamma} \right) \\
&\le c 2^m (2^m \rho(x,z))^{\mu(x,z,2^{-m})k} \\
&\quad \times \left(\left(\frac{2^{-m}}{2^{-m} + \rho(x,y)} \right)^{1+\delta+\mu(x,z,2^{-m})k} + \left(\frac{2^{-m}}{2^{-m} + \rho(y,z)} \right)^{1+\delta+\mu(x,z,2^{-m})k} \right) \\
&= c \rho(x,z)^{\mu(x,z,2^{-m})k} \left(\frac{2^{-m\delta}}{(2^{-m} + \rho(x,y))^{1+\delta+\mu(x,z,2^{-m})k}} + \frac{2^{-m\delta}}{(2^{-m} + \rho(y,z))^{1+\delta+\mu(x,z,2^{-m})k}} \right).
\end{aligned}$$

Property (iii) is proved similarly. Finally, we prove the polynomial reproduction property (iv). By (3.12) for any $P \in \Pi_{r-1}$, there exist coefficients $\{c_\lambda\}_{\lambda \in \Lambda_m}$ such that $P = \sum_{\lambda \in \Lambda_m} c_\lambda \varphi_\lambda$. For fixed $y \in \mathbb{R}^n$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} S_n(x,y) P(x)\, dx &= \int_{\mathrm{R}^n} \Big(\sum_{\lambda, \lambda' \in \Lambda_m} B_{\lambda,\lambda'} \varphi_\lambda(x) \varphi_{\lambda'}(y) \Big) \Big(\sum_{\lambda'' \in \Lambda_m} c_{\lambda''} \varphi_{\lambda''}(x) \Big) dx \\
&= \sum_{\lambda, \lambda', \lambda'' \in \Lambda_m} c_{\lambda''} B_{\lambda,\lambda'} \varphi_{\lambda'}(y) \int_{\mathbb{R}^n} \varphi_\lambda(x) \varphi_{\lambda''}(x)\, dx \\
&= \sum_{\lambda', \lambda'' \in \Lambda_m} c_{\lambda''} \varphi_{\lambda'}(y) \sum_{\lambda \in \Lambda_m} B_{\lambda,\lambda'} A_{\lambda'',\lambda} \\
&= \sum_{\lambda', \lambda'' \in \Lambda_m} c_{\lambda''} \varphi_{\lambda'}(y) \delta_{\lambda',\lambda''} = \sum_{\lambda'' \in \Lambda_m} c_{\lambda''} \varphi_{\lambda''}(y) = P(y).
\end{aligned}$$

The proof that $P(x) = \int_{\mathbb{R}^n} S_n(x,y) P(y)\, dy$ is similar. This concludes the proof of the corollary.

♢

4. Construction of anisotropic wavelet frames

4.1. **Wavelet operators.** Let $\{S_m\}_{m\in\mathbb{Z}}$ be a multiresolution analysis of order (μ,δ,r). Then it is clear that the kernels of the *wavelet operators* $D_m := S_{m+1}-S_m$, satisfy properties (i)-(iii) of Definition 3.1, while the *polynomial reproduction* property (iv) is replaced with the *zero moments* property

$$\int_{\mathbb{R}^n} D_m(x,y)\,P(y)\,dy = 0, \quad \int_{\mathbb{R}^n} D_m(x,y)\,P(x)\,dx = 0, \tag{4.1}$$

for every polynomial $P\in\Pi_{r-1}$.

We now show that two wavelet operators (kernels) from different scales are 'almost orthogonal'.

Lemma 4.1. *Assume that two kernels operators $\{D_m^1\}$ and $\{D_m^2\}$, $m\in\mathbb{Z}$, satisfy* (4.1) *for $r\geq 1$ and conditions $(i)-(ii)$ of a multiresolution with order $(\mu,\delta+\mu_1 r,r)$ for some $\delta\geq\mu_1 r$. Then*

$$\left|D_k^1 D_l^2(x,y)\right| \leq c2^{-|k-l|\mu_0 r}\frac{2^{-\min(k,l)\delta}}{\left(2^{-\min(k,l)}+\rho(x,y)\right)^{1+\delta}}, \quad k,l\in\mathbb{Z}. \tag{4.2}$$

Proof. For simplicity of notation, assume that $\{D_m\}=\{D_m^1\}=\{D_m^2\}$. The proof of the general case is similar. The kernel of the operator D_kD_l is

$$D_kD_l(x,y) = \int_{\mathbb{R}^d} D_k(x,z)\,D_l(z,y)\,dz.$$

Assume that $l\leq k$. The proof for the case $k<l$ is similar. We apply the zero-moment property (4.1) to obtain

$$\begin{aligned}
|D_kD_l(x,y)| &= \left|\int_{\mathbb{R}^n} D_k(x,z)\,D_l(z,y)\,dz\right| \leq \int_{\mathbb{R}^n} |D_k(x,z)|\,|R_{r,x}(D_l(\cdot,y))(z)|\,dz \\
&\leq \int_{\rho(x,z)\leq\frac{1}{2\kappa}(2^{-l}+\rho(x,y))} |D_k(x,z)|\,|R_{r,x}(D_l(\cdot,y))(z)|\,dz \\
&\quad + \int_{\rho(x,y)\leq\rho(y,z)} |D_k(x,z)|\,|R_{r,x}(D_l(\cdot,y))(z)|\,dz \\
&\quad + \int_{\rho(x,y)>\rho(y,z),\rho(x,z)>\frac{1}{2\kappa}(2^{-l}+\rho(x,y))} |D_k(x,z)|\,|R_{r,x}(D_l(\cdot,y))(z)|\,dz \\
&=: I+II+III.
\end{aligned}$$

We estimate each of the three integrals separately. Applying the properties of the kernels and (3.2) we derive

$$\begin{aligned}
I &= \int_{\rho(x,z)\le\frac{1}{2\kappa}(2^{-l}+\rho(x,y))} |D_k(x,z)|\,|R_{r,x}(D_l(\cdot,y))(z)|\,dz\\
&\le c\int_{\rho(x,z)\le\frac{1}{2\kappa}(2^{-l}+\rho(x,y))} \frac{2^{-k\delta}}{(2^{-k}+\rho(x,z))^{1+\delta}}\rho(x,z)^{\mu_0 r}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\mu_0 r}}dz\\
&\le c2^{-k\delta}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\mu_0 r}}\int_{\mathbb{R}^n}\frac{\rho(x,z)^{\mu_0 r}}{(2^{-k}+\rho(x,z))^{1+\delta}}dz\\
&\le c2^{-k\delta}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\mu_0 r}}2^{k(\delta-\mu_0 r)}\\
&\le c2^{(l-k)\mu_0 r}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta}}.
\end{aligned}$$

The estimate of the second integral is similar to first, only here we use property (ii) in Definition 3.1 the fact that $\rho(x,y)\le\rho(y,z)$

$$\begin{aligned}
II &= \int_{\rho(x,y)\le\rho(y,z)} |D_k(x,z)|\,|R_{r,x}(D_l(\cdot,y))(z)|\,dz\\
&\le c\int_{\rho(x,y)\le\rho(y,z)} \frac{2^{-k\delta}}{(2^{-k}+\rho(x,z))^{1+\delta}}\rho(x,z)^{\mu(x,z,2^{-l})r}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\mu(x,z,2^{-l})r}}dz\\
&\le 2^{-k\delta}\left(\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\mu_0 r}}\int_{\rho(x,z)\le 2^{-l}}\frac{\rho(x,z)^{\mu_0 r}}{(2^{-k}+\rho(x,z))^{1+\delta}}dz\right.\\
&\qquad\left.+\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\mu_1 r}}\int_{\rho(x,z)>2^{-l}}\frac{\rho(x,z)^{\mu_1 r}}{(2^{-k}+\rho(x,z))^{1+\delta}}dz\right)\\
&\le 2^{-k\delta}\left(\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\mu_0 r}}2^{k(\delta-\mu_0 r)}+\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta+\mu_1 r}}2^{k(\delta-\mu_1 r)}\right)\\
&\le c2^{(l-k)\mu_0 r}\frac{2^{-l\delta}}{(2^{-l}+\rho(x,y))^{1+\delta}}.
\end{aligned}$$

We proceed with the estimate of III. Observe that the integration domain in this term satisfies $\rho(x,z)>2^{-l}/2\kappa$ which implies that we can assume $\mu\left(x,z,2^{-l}\right)=\mu_1$

$$\begin{aligned}
III &= \int_{\rho(x,y)>\rho(y,z),\ \rho(x,z)>\frac{1}{2\kappa}\left(2^{-l}+\rho(x,y)\right)} \left|D_k\left(x,z\right)\right| \left|R_{r,x}\left(D_l\left(\cdot,y\right)\right)(z)\right| dz \\
&\leq c\int_{\rho(x,z)>\frac{1}{2\kappa}\left(2^{-l}+\rho(x,y)\right)} \left|D_k\left(x,z\right)\right| \rho\left(x,z\right)^{\mu_1 r} \frac{2^{-l\delta}}{\left(2^{-l}+\rho\left(z,y\right)\right)^{1+\delta+\mu_1 r}} dz \\
&\leq c2^{-l\delta}\left(\int_{\rho(x,z)>\frac{1}{2\kappa}\left(2^{-l}+\rho(x,y)\right)} \frac{2^{-k\delta}}{\left(2^{-k}+\rho\left(x,z\right)\right)^{1+\delta}} \frac{\rho\left(z,y\right)^{\mu_1 r}}{\left(2^{-l}+\rho\left(z,y\right)\right)^{1+\delta+\mu_1 r}} dz\right. \\
&\qquad \left. + \int_{\rho(x,z)>\frac{1}{2\kappa}\left(2^{-l}+\rho(x,y)\right)} \frac{2^{-k(\delta+\mu_1 r)}}{\left(2^{-k}+\rho\left(x,z\right)\right)^{1+\delta+\mu_1 r}} \frac{\rho\left(x,y\right)^{\mu_1 r}}{\left(2^{-l}+\rho\left(z,y\right)\right)^{1+\delta+\mu_1 r}} dz\right) \\
&\leq c2^{-k\delta} \frac{2^{-l\delta}}{\left(2^{-l}+\rho\left(x,y\right)\right)^{1+\delta}} \int_{\mathbb{R}^n} \frac{\rho\left(z,y\right)^{\mu_1 r}}{\left(2^{-l}+\rho\left(z,y\right)\right)^{1+\delta+\mu_1 r}} dz \\
&\qquad + 2^{-k(\delta+\mu_1 r)} \frac{2^{-l\delta}}{\left(2^{-l}+\rho\left(x,y\right)\right)^{1+\delta+\mu_1 r}} \int_{\mathbb{R}^n} \frac{\rho\left(x,y\right)^{\mu_1 r}}{\left(2^{-l}+\rho\left(z,y\right)\right)^{1+\delta+\mu_1 r}} dz \\
&\leq c2^{-k\delta} \frac{2^{-l\delta}}{\left(2^{-l}+\rho\left(x,y\right)\right)^{1+\delta}} 2^{l\delta} + c2^{-k(\delta+\mu_1 r)} \frac{2^{-l\delta}}{\left(2^{-l}+\rho\left(x,y\right)\right)^{1+\delta+\mu_1 r}} \rho\left(x,y\right)^{\mu_1 r} 2^{l(\delta+\mu_1 r)} \\
&\leq c2^{(l-k)\mu_0 r} \frac{2^{-l}}{\left(2^{-l}+\rho\left(x,y\right)\right)^{1+\delta}}.
\end{aligned}$$

$\diamondsuit$

4.2. **Dual wavelet operators.** In the previous section we defined the wavelet (difference) operators and reviewed some of their properties. In this section we leverage significantly on the results of Han and Sawyer [22] concerning the Calderón reproducing formula in spaces of homogeneous type and adapt them to our special setting. We begin with the definitions for anisotropic test functions and molecules.

Definition 4.2. Fix a quasi-distance ρ on $\mathbb{R}^n$. A function $f \in C\left(\mathbb{R}^n\right)$ belongs to the *anisotropic test function space* $\mathcal{M}\left(\varepsilon,\delta,x_0,t\right)$, $0<\varepsilon,\delta\leq 1$, $x_0\in\mathbb{R}^n$, $t\in\mathbb{R}$ if there exists a constant C such that

(i) $\left|f\left(x\right)\right| \leq C\frac{2^{-t\delta}}{\left(2^{-t}+\rho(x,x_0)\right)^{1+\delta}}$, $\forall x\in\mathbb{R}^n$.

(ii) $\left|f\left(x\right)-f\left(y\right)\right| \leq C\rho\left(x,y\right)^{\varepsilon}\frac{2^{-t\delta}}{\left(2^{-t}+\rho(x,x_0)\right)^{1+\delta+\varepsilon}}$ for all $x,y\in\mathbb{R}^n$,

where $\rho\left(x,y\right)\leq\frac{1}{2\kappa}\left(2^{-t}+\rho\left(x,x_0\right)\right)$, with κ defined in (2.3).

One can easily show that $\mathcal{M}\left(\varepsilon,\delta,x_0,t\right)$ is a Banach space with $\|f\|_{\mathcal{M}}$ defined by the infimum over all constants C satisfying (i) and (ii). We also denote $\mathcal{M}\left(\varepsilon,\delta\right):=\mathcal{M}\left(\varepsilon,\delta,0,0\right)$.

Definition 4.3. An anisotropic test function $f\in\mathcal{M}\left(\varepsilon,\delta,x_0,t\right)$ is said to be a *molecule* in $\mathcal{M}_0\left(\varepsilon,\delta,r,x_0,t\right)$ if

$$\int_{\mathbb{R}^n} f\left(y\right)dy = 0.$$

As Y. Meyer pointed out the Banach space $C^{\varepsilon}\left(\mathbb{R}\right)$ of Hölder functions with exponent β has the following properties:

(1) If $0<\varepsilon<1$, $C^{\varepsilon}\left(\mathbb{R}\right)$ is isomorphic to $l^{\infty}\left(\mathbb{Z}\right)$,
(2) If $\varepsilon=1$, the Zygmond class is isomorphic to $L_{\infty}\left(\mathbb{R}\right)$.

It implies that the dual space of $C^{\varepsilon}(\mathbb{R})$ is not a functional space. Indeed, the dual space of $l^{\infty}(\mathbb{Z})$ is not a sequence space. This remark also applies to $\mathcal{M}(\varepsilon,\delta)$ and its dual space of *anisotropic distributions* $\mathcal{M}'(\varepsilon,\delta)$.Of course, this can be solved. It suffices to replace C^{ε} by the closure in the C^{ε} norm of C^{γ} for some $\gamma>\varepsilon$. This closure does not depend on γ. For this purpose, we denote by $\overset{\circ}{\mathcal{M}}(\varepsilon,\delta)$ the closure of $\mathcal{M}(\gamma,\delta)$ in the norm of $\mathcal{M}(\varepsilon,\delta)$. Then, we define the $\overset{\circ}{\mathcal{M}}'(\varepsilon,\delta)$ as the dual of $\overset{\circ}{\mathcal{M}}(\varepsilon,\delta)$.

We are now ready to state the Calderón reproducing formula which implies the existence of wavelet dual operators.

Theorem 4.4. [Continuous Calderón reproducing formula] *Let $(\mathbb{R}^n,\rho,dx)$ be a normal space of homogeneous type and let $\{S_m\}$ be an anisotropic multiresolution of order (μ,δ,r) with respect to ρ. For $D_m:=S_{m+1}-S_m$ there exist linear operators $\{\tilde{D}_m\}_{m\in\mathbb{Z}}$ and $\{\hat{D}_m\}_{m\in\mathbb{Z}}$ such that for all $f\in\mathcal{M}_0(\varepsilon,\gamma)$, $0<\varepsilon,\gamma<\mu_0$*

$$f(x)=\sum_{m\in\mathbb{Z}}\tilde{D}_mD_m(f)(x)=\sum_{m\in\mathbb{Z}}D_m\hat{D}_m(f)(x), \tag{4.3}$$

where the series converges in the norm $\mathcal{M}(\varepsilon',\gamma')$, $\varepsilon'<\varepsilon$, $\gamma'<\gamma$, and in the space $L_p(\mathbb{R}^n)$, $1<p<\infty$. Furthermore, for any $\varepsilon<\mu_0$, the kernels of $\{\tilde{D}_m\}_{m\in\mathbb{Z}}$ and $\{\hat{D}_m\}_{m\in\mathbb{Z}}$ satisfy the conditions $(i)-(iii)$ of multiresolution of order $(\mu,\varepsilon,1)$ (with constants that depend on ε) and the rth zero moments conditions (4.1) *for r.*

By the duality argument we obtain

Corollary 4.5. *Under the conditions of Theorem 4.4, the series in* (4.3) *converges in $\overset{\circ}{\mathcal{M}}'(\varepsilon_*,\delta_*)$ with $\varepsilon<\varepsilon_*<\mu_0$, $\gamma<\gamma_*<\mu_0$, whenever $\overset{\circ}{\mathcal{M}}(\varepsilon,\gamma)$.*

Proof of Theorem 4.4. The method of proof is essentially similar to the method of [22]. We use Coifman's idea and write the identity operator I by

$$I=\sum_k D_k=\sum_k D_k\sum_l D_l=\sum_{k,l}D_kD_l.$$

We define for some integer $N>0$, the operator $D_m^N:=\sum_{|j|\le N}D_{m+j}$ and the operators T_N and R_N by

$$I=\sum_{k,l}D_kD_l=\sum_{k\in\mathbb{Z}}D_k^ND_k+\sum_{|j|>N}\sum_{k\in\mathbb{Z}}D_{k+j}D_k=:T_N+R_N.$$

Let $0<\varepsilon,\gamma<\mu_0$. We claim that R_N is bounded on $\mathcal{M}_0(\varepsilon,\gamma,x_0,t)$ for any $x_0\in\mathbb{R}^n$, $t\in\mathbb{R}$. Moreover, there exist constants $\tau>0$ and $C>0$ such that for $f\in\mathcal{M}_0(\varepsilon,\gamma,x_0,t)$

$$\|R_Nf\|_{\mathcal{M}_0(\varepsilon,\gamma,x_0,t)}\le C2^{-N\tau}\|f\|_{\mathcal{M}_0(\varepsilon,\gamma,x_0,t)}. \tag{4.4}$$

Assume the claim for a moment. Choosing a large N such that $C2^{-N\tau}<1$, then (4.4) implies that the operator T_N^{-1} exists and is bounded on $\mathcal{M}_0(\varepsilon,\gamma,x_0,t)$. Thus, we obtain $I=T_N^{-1}T_N=\sum_m\left(T_N^{-1}D_m^N\right)D_m=\sum_m\tilde{D}_mD_m$, where $\tilde{D}_m:=T_N^{-1}D_m^N$.

The regularity conditions of the kernels $\{D_m\}$ and (4.1) imply that for any fixed N and $y\in\mathbb{R}^n$ the function $D_m^N(\cdot,y)$ is in $\mathcal{M}_0(\mu_0,\delta)$. This gives that $\tilde{D}_m(\cdot,y)=T_N^{-1}D_m^N(\cdot,y)$ is in $\mathcal{M}_0(\varepsilon,\gamma)$ for any $0<\varepsilon,\gamma<\mu_0$. Similarly, we may write

$$I=T_NT_N^{-1}=(\sum_m D_m^ND_m)T_N^{-1}=\sum_m D_mD_m^NT_N^{-1}=\sum_m D_m\hat{D}_m,$$

where $\hat{D}_m := D_m^N T_N^{-1}$.

By the same reasons, for any fixed N and $x \in \mathbb{R}^n$, the function $\hat{D}_m(x,\cdot)$ is in $\mathcal{M}_0(\varepsilon,\gamma)$ for any $0<\varepsilon,\gamma<\mu_0$.

$\diamondsuit$

Discussion. In Theorem 4.4 we apply tools from the general theory of spaces of homogeneous type to construct dual wavelet operators. Although the kernels of the dual operators $\{\tilde{D}_m\}$ and $\{\hat{D}_m\}$ have the same number of r vanishing moments as $\{D_m\}$, we only claim very 'modest' regularity and size conditions for them. For example, in Theorem 4.4 we claim that for any $0<\gamma<\mu_0$, there exists a constant $C>0$ (that also depends on γ) such that

$$|\tilde{D}_m(x,y)|, |\hat{D}_m(x,y)| \le C\frac{2^{-m\gamma}}{\left(2^{-m}+\rho(x,y)\right)^{1+\gamma}},$$

while in contrast, the construction of the anisotropic multiresolution over the ellipsoid cover in Section 3 gives wavelet kernels $\{D_m\}$ that satisfy for any positive $\delta>0$ (see Corollary 3.10)

$$|D_m(x,y)| \le C\frac{2^{-m\delta}}{\left(2^{-m}+\rho(x,y)\right)^{1+\delta}}.$$

It is still not known to us how to correctly define higher order anisotropic test function spaces and prove that the operator $R_N := \sum\limits_{|j|>N}\sum\limits_{k\in\mathbb{Z}} D_{k+j}D_k$ is a bounded operator on these higher order spaces as in (4.4).

As in [20], we may apply the Calderón reproducing formula to obtain the following Littlewood-Paley result

Proposition 4.6. *Let $\{S_m\}_{m\in\mathbb{Z}}$ be any anisotropic multiresolution and let $D_m = S_{m+1} - S_m$. Then for all $f \in L_p(\mathbb{R}^n)$, $1<p<\infty$, we have*

$$\|f\|_p \sim \left\|\left(\sum_m |D_m(f)(\cdot)|^2\right)^{1/2}\right\|_p.$$

4.3. **Discrete wavelet frames.** We construct wavelet frames using the discrete Calderón reproducing formula, which in turn is obtained by 'sampling' the continuous Calderón reproducing formula. First we introduce the following sampling process.

Definition 4.7. Let ρ be a quasi-distance on $\mathbb{R}^n$. We call a set of closed domains $\{\Omega_{m,k}\}$, $m\in\mathbb{Z}$, $k\in I_m$, and points $y_{m,k}\in\Omega_{m,k}$, a *sampling set* if it satisfies the following properties:

(a) For each $m\in\mathbb{Z}$, the sets $\Omega_{m,k}$, $k\in I_m$, are pairwise interior disjoint.

(b) For all $m\in\mathbb{Z}$, $\mathbb{R}^n = \bigcup\limits_{k\in I_m}\Omega_{m,k}$.

(c) Each set $\Omega_{m,k}$ satisfies $\Omega_{m,k}\subset B(x_{m,k}, c2^{-m})$ for some point $x_{m,k}\in\mathbb{R}^n$ and fixed $c>0$ (here the ball corresponds to ρ).

(d) There exists a constant $c'>0$ such that for any $m\in\mathbb{Z}$ and $k\in I_m$, we have that $\rho(y_{m,k}, y_{m,k'}) > c'2^{-m}$ for all $k'\in I_m$, $k'\ne k$, except perhaps for a bounded set.

Examples

(1) One can construct a sampling set from an ellipsoid cover. We begin by picking a maximal set of disjoint ellipsoids as follows: For each level Θ_m we enumerate the ellipsoids as $\theta_{m,j}$, $j\ge 1$. We define $\theta'_{m,1} := \theta_{m,1}$ and then inductively for $k,j>1$, $\theta'_{m,k} := \theta_{m,j}$ if $\mathrm{int}\left(\left(\bigcup_{i=1}^{k-1}\theta'_{m,i}\right)\cap\theta_{m,j}\right)=\emptyset$. We also select $y_{m,k}$ as the center of $\theta'_{m,k}$. After this step, the domains $\{\theta'_{m,k}\}$ and sampling points $\{y_{m,k}\}$ satisfy properties

(a), (c) and (d), but not (b). To see that property (d) is indeed satisfied, denote by $\sigma_{\max}(\theta)$ the length of the maximal semi-axis of any ellipsoid θ. If two ellipsoids $\theta'_{m,k}$ and $\theta'_{m,k'}$ do not intersect then evidently $|y_{m,k} - y_{m,k'}| > \sigma_{\max}(\theta_{m,k})$. Let η be any ellipsoid at the level $m' > m$ such that $y_{m,k} \in \eta$. By [13, Lemma 2.2] we have that $\sigma_{\max}(\eta) \le a_5 2^{-(m'-m)a_6} \sigma_{\max}(\theta_{m,k})$. Therefore, if $m' > m + \nu$, $\nu := \lceil a_6^{-1} \log_2 a_5 \rceil$, then $y_{m,k'} \notin \eta$. We conclude that $\rho(y_{m,k}, y_{m,k'}) > a_1 2^{-(m+\nu)} = c' 2^{-m}$. Observe that each $\theta_{m,j}$ that was not selected at the previous step, must intersect one of the ellipsoids $\theta'_{m,k}$. We now denote $\Omega_{m,k} := \theta'_{m,k}$ and iterate on the ellipsoids that were not selected. For each such ellipsoid $\theta_{m,j}$ we add the domain $\theta_{m,j} - \left(\bigcup_{i=1}^{\infty} \Omega_{m,i}\right)$ (if not empty at this stage) to one of the domains $\Omega_{m,k}$ only if $\theta_{m,j}$ intersects $\theta'_{m,k}$. Observe that the domains $\Omega_{m,k}$ are possibly enlarged during this process, but this is controlled by the fact that each ellipsoid $\theta'_{m,k}$ has no more than N_1 neighbors. Evidently, we attain domains $\{\Omega_{m,k}\}$ that satisfy all of the conditions.

(2) Christ's 'dyadic cube' construction for spaces of homogeneous type [6] also satisfies the above conditions. As the name suggests, it has similar properties to the regular, isotropic dyadic cube cover of $\mathbb{R}^n$. For example, each sampling 'cube' $\Omega_{m+1,k}$ is contained in one and only one sampling 'cube' $\Omega_{m,k'}$ for some $k' \in I_m$. Also, each sampling domain at the level m is 'substantial' in the sense that it contains a ball of radius $\ge c' 2^{-m}$. Therefore property (d) is satisfied, provided the sampling points $y_{m,k} \in \Omega_{m,k}$ are picked from these inner balls.

Theorem 4.8. *Let $\{S_m\}_{m\in\mathbb{Z}}$ be an anisotropic multiresolution of order (μ, δ, r), with respect to the quasi-distance induced by an ellipsoid cover Θ. Denote $D_m := S_{m+1} - S_m$ and let $\{\Omega_{m,k}\}$ and $\{y_{m,k}\}$, $y_{m,k} \in \Omega_{m,k}$ be a sampling set for Θ. Then there exists $N > 0$ and linear operators $\{\hat{E}_m\}$ such that for all $f \in \mathcal{M}_0(\varepsilon, \gamma)$, $0 < \varepsilon, \gamma < \mu_0$,*

$$f(x) = \sum_{m\in\mathbb{Z}} \sum_{k\in I_{m+N}} |\Omega_{m+N,k}| \, \hat{E}_m(f)(y_{m+N,k}) \, D_m(x, y_{m+N,k}), \tag{4.5}$$

where the convergence is in $\mathcal{M}(\varepsilon', \gamma')$, $\varepsilon' < \varepsilon$, $\gamma' < \gamma$, and in the space $L_p(\mathbb{R}^n)$, $1 < p < \infty$. Furthermore, the kernels of $\{\hat{E}_m\}$ satisfy conditions (i)-(iii) of anisotropic multiresolution with $(\mu, \varepsilon, 1)$ for all $\varepsilon < \mu_0$ (with constants that depend on ε) and the zero moments conditions (4.1) *for r.*

Sketch of proof. The proof is similar to the proof in [21]. The discrete formula (4.5) is obtained from the continuous formula (4.3) as follows. We fix some $N > 0$ and apply (4.3) to obtain for $f \in \mathcal{M}_0(\varepsilon, \gamma)$

$$\begin{aligned}
f(x) &= \sum_m D_m \hat{D}_m(f)(x) \\
&= \sum_m \sum_{k\in I_{m+N}} \int_{\Omega_{m+N,k}} D_m(x,y) \, \hat{D}_m(f)(y) \, dy \\
&= \sum_m \sum_{k\in I_{m+N}} |\Omega_{m+N,k}| \, D_m(x, y_{m+N,k}) \, \hat{D}_m(f)(y_{m+N,k}) \\
&\quad + \Bigg\{ \sum_m \sum_{k\in I_{m+N}} \int_{\Omega_{m+N,k}} [D_m(x,y) - D_m(x, y_{m+N,k})] \, \hat{D}_m(f)(y) \, dy \\
&\qquad + \sum_m \sum_{k\in I_{m+N}} \int_{\Omega_{m+N,k}} D_m(x, y_{m+N,k}) \left[\hat{D}_m(f)(y) - \hat{D}_m(f)(y_{m+N,k})\right] dy \Bigg\} \\
&=: \tilde{T}_N(x) + \tilde{R}_N(x).
\end{aligned}$$

It is shown in [21] that for sufficiently large $N > 0$, the operator $\tilde{R}_N$ is bounded on $\mathcal{M}_0(\varepsilon, \gamma)$ and its norm is strictly smaller than 1. Therefore, there exists the inverse operator $\tilde{T}_N^{-1}$ and it is bounded on $\mathcal{M}_0(\varepsilon, \gamma)$. Thus, with $\hat{E}_m := \hat{D}_m \tilde{T}_N^{-1}$ we get

$$\begin{aligned} f(x) &= \tilde{T}_N \tilde{T}_N^{-1}(f)(x) \\ &= \sum_m \sum_{k \in I_{m+N}} |\Omega_{m+N,k}| \, D_m(x, y_{m+N,k}) \hat{D}_m \left(\tilde{T}_N^{-1}(f)\right)(y_{m+N,k}) \\ &= \sum_m \sum_{k \in I_{m+N}} |\Omega_{m+N,k}| \, D_m(x, y_{m+N,k}) \hat{E}_m(f)(y_{m+N,k}) \,. \end{aligned}$$

$\diamondsuit$

Denoting the index set $K_m := I_{m+N}$, the functions $\{\psi_{m,k}\}$ by

$$\psi_{m,k}(x) := |\Omega_{m+N,k}|^{1/2} D_m(x, y_{m+N,k})$$

and the functionals $\left\{\tilde{\psi}_{m,k}\right\}$ by $\tilde{\psi}_{m,k}(x) := |\Omega_{m+N,k}|^{1/2} \hat{E}_m(y_{m+N,k}, x)$, $m \in \mathbb{Z}$, $k \in K_m$, we obtain the following representation

$$f(x) = \sum_m \sum_{k \in K_m} \langle f, \tilde{\psi}_{m,k} \rangle \psi_{m,k}(x). \tag{4.6}$$

Observe that the anisotropic wavelet representation (4.6) resembles a classical isotropic wavelet representation (see [9, 14, 18]). However, here the wavelets are specifically 'tuned' to the geometry of the given ellipsoid cover and the induced quasi-distance. Lastly, we show that the anisotropic wavelets constitute a frame (see Definition 1.1)

Theorem 4.9. *Let $\{S_m\}_{m \in \mathbb{Z}}$ be an anisotropic multiresolution of order (μ, δ, r). Denote $D_m := S_{m+1} - S_m$ and let $\{\Omega_{m,k}\}$ and $\{y_{m,k}\}$, $y_{m,k} \in \Omega_{m,k}$ be a sampling set for Θ. If $r > \mu_0^{-1}$, then there exists constants $0 < A \le B < \infty$ such that such that for any $f \in L_2(\mathbb{R}^n)$*

$$A \|f\|_2^2 \le \sum_m \sum_{k \in K_m} |\langle f, \tilde{\psi}_{m,k} \rangle|^2 \le B \|f\|_2^2 \,. \tag{4.7}$$

Proof. The proof is similar to the proof of [21, Theorem 3.35]. We begin with a proof of the left hand side of (4.7). An identical argument to the one used to prove Lemma 4.1 yields

$$\begin{aligned} |\langle \psi_{m,k}, \psi_{m',k'} \rangle| &= |\Omega_{m+N,k}|^{1/2} |\Omega_{m'+N,k'}|^{1/2} \left| \int_{\mathbb{R}^n} D_m(x, y_{m+N,k}) \, D_{m'}(x, y_{m'+N,k'}) \, dx \right| \\ &\le c \, |\Omega_{m+N,k}|^{1/2} |\Omega_{m'+N,k'}|^{1/2} 2^{-|m-m'|\mu_0 r} \frac{2^{-\min(m,m')\delta}}{\left(2^{-\min(m,m')} + \rho(y_{m+N,k}, y_{m'+N,k'})\right)^{1+\delta}} \\ &\le c 2^{-|m-m'|\mu_0 r} \left(\frac{2^{-\min(m,m')}}{2^{-\min(m,m')} + \rho(y_{m+N,k}, y_{m'+N,k'})} \right)^{1+\delta} . \end{aligned}$$

We denote $\omega(m,k) = 2^{-m}$ and apply this last estimate, condition (d) of Definition 4.7 and the condition $r > \mu_0^{-1}$ to compute for fixed $m \in \mathbb{Z}$, $k \in K_m$,

$$\begin{aligned}
&\sum_{m',k'} |\langle \psi_{m,k}, \psi_{m',k'} \rangle| \, \omega(m',k') \\
&\quad \le c \sum_{m',k'} 2^{-m'} 2^{-|m-m'|\mu_0 r} \Big(\frac{2^{-\min(m,m')}}{2^{-\min(m,m')} + \rho(y_{m+N,k}, y_{m'+N,k'})} \Big)^{1+\delta} \\
&\quad \le c \sum_{m'} 2^{-m'} 2^{-|m-m'|\mu_0 r} 2^{m'} \sum_{k'} 2^{-m'} \Big(\frac{2^{-\min(m,m')}}{2^{-\min(m,m')} + \rho(y_{m+N,k}, y_{m'+N,k'})} \Big)^{1+\delta} \\
&\quad \le c \sum_{m'} 2^{-m'} 2^{-|m-m'|\mu_0 r} 2^{m'} 2^{-\min(m,m')} \\
&\quad \le c \Big(\sum_{m' \le m} 2^{-m'} 2^{-(m-m')\mu_0 r} + \sum_{m' > m} 2^{-m'} 2^{-(m'-m)\mu_0 r} 2^{m'} 2^{-m} \Big) \\
&\quad \le c \Big(2^{-m} \sum_{m' \le m} 2^{-(m-m')(\mu_0 r - 1)} + 2^{-m} \sum_{m' > m} 2^{-(m'-m)\mu_0 r} \Big) \le c\omega(m,k).
\end{aligned}$$

The above estimate is exactly the condition of Schur's Lemma (see [28, Section 8.4] for the case of isotropic dyadic cubes and wavelets) which we use here to show that the infinite matrix $A := \{\langle \psi_{m,k}, \psi_{m',k'} \rangle\}$ is bounded on l_2 sequences over the 'sampling' index space. In particular, for the sequence $\alpha := \{\langle f, \tilde{\psi}_{m,k} \rangle\}_{m \in \mathbb{Z}, k \in K_m}$ we obtain

$$\|f\|_2^2 = \langle A\alpha, \alpha \rangle \le \|A\| \, \|\alpha\|^2 \le c \sum_{m,k} |\langle f, \tilde{\psi}_{m,k} \rangle|^2.$$

Next we prove the right hand side inequality of (4.7). By definition we have

$$\begin{aligned}
\sum_m \sum_{k \in K_m} |\langle f, \tilde{\psi}_{m,k} \rangle|^2 &= \sum_m \sum_{k \in K_m} |\Omega_{m+N,k}| \, |\hat{E}_m(f)(y_{m+N,k})|^2 \\
&= \sum_m \sum_{k \in K_m} \int_{\Omega_{m+N,k}} |\hat{E}_m(f)(y_{m+N,k})|^2 dy.
\end{aligned}$$

Theorem 4.4 shows that there exist operators $\{\hat{D}_m\}_{m \in \mathbb{Z}}$ that satisfy the regularity conditions (i)-(iii) of multiresolution of order $(\mu, \varepsilon, 1)$, $\varepsilon < \mu_0$, have r zero moments and for which $f = \sum_m \tilde{D}_m D_m(f)$. One can show (using a similar, but simpler, approach to the proof of (4.2)) that for $m, j \in \mathbb{Z}$

$$|\hat{E}_m \tilde{D}_j(x,y)| \le c 2^{|m-j|\varepsilon} \frac{2^{-\min(m,j)\varepsilon}}{\big(2^{-\min(m,j)} + \rho(x,y)\big)^{1+\varepsilon}}. \tag{4.8}$$

It is well known (see e.g. [29]) that in the setting of spaces of homogeneous type the *Maximal function* defined by

$$Mf(x) := \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| \, dy,$$

where the sup is over all anisotropic balls B, is bounded on L_p, $1 < p \le \infty$, i.e.

$$\|Mf\|_p \le c \|f\|_p, \quad f \in L_p. \tag{4.9}$$

We use the Continuous Calderón formula and (4.8) to estimate each coefficient

$$
\begin{aligned}
&|\langle f, \tilde{\psi}_{m,k}\rangle|^2 = \int_{\Omega_{m+N,k}} |\hat{E}_m(f)(y_{m+N,k})|^2 dy \\
&= \int_{\Omega_{m+N,k}} \Big|\sum_j \hat{E}_m \tilde{D}_j D_j\,(f)\,(y_{m+N,k})\Big|^2 dy \\
&\le c \int_{\Omega_{m+N,k}} \Big(\sum_j \int_{\mathbb{R}^n} 2^{|m-j|\varepsilon} \frac{2^{-\min(m,j)\varepsilon}}{\left(2^{-\min(m,j)} + \rho\left(y_{m+N,k}, z\right)\right)^{1+\varepsilon}} \left|D_j\,(f)\,(z)\right| dz\Big)^2 dy \\
&\le c \int_{\Omega_{m+N,k}} \Big(\sum_j 2^{|m-j|\varepsilon} M D_j\,(f)\,(y)\Big)^2 dy.
\end{aligned}
$$

Applying the discrete Holder inequality and the Maximal inequality (4.9) give

$$
\begin{aligned}
\sum_m \sum_{k\in K_m} |\langle f, \tilde{\psi}_{m,k}\rangle|^2 &\le c \sum_m \int_{\mathbb{R}^n} \Big(\sum_j 2^{|m-j|\varepsilon} M D_j\,(f)\,(y)\Big)^2 dy \\
&\le c \sum_m \int_{\mathbb{R}^n} \Big(\sum_j 2^{|m-j|\varepsilon}\Big)\Big(\sum_j 2^{|m-j|\varepsilon} \left(M D_j\,(f)\,(y)\right)^2\Big) dy \\
&\le c \sum_j \|M D_j\,(f)\|_2^2 \\
&\le c \sum_j \|D_j\,(f)\|_2^2 \le c\,\|f\|_2^2 .
\end{aligned}
$$

$\diamondsuit$

4.4. **Two level split frames.** Following [13] we introduce a useful representation for the wavelet kernels $D_m\,(x,y)$ using the 'two level split' construction. Denote

$$
\mathrm{M}_m := \{\nu = (\eta, \theta, \beta) \,:\, \eta \in \Theta_{m+1}, \theta \in \Theta_m, \eta \cap \theta \neq \emptyset, |\beta| < r\}, \quad m \in \mathbb{Z},
$$

and define using (3.7) and (3.9)

$$
F_\nu := P_{\eta,\beta} \tilde{\phi}_\eta \tilde{\phi}_\theta = \varphi_{\eta,\beta} \tilde{\phi}_\theta, \quad \nu \in \mathrm{M}_m, \tag{4.10}
$$

We also denote $\mathcal{F}_m := \{F_\nu \,:\, \nu \in \mathrm{M}_m\}$ and set $W_m := span\,(\mathcal{F}_m)$.

Note that $F_\nu \in C^\infty$, $\mathrm{supp}\,(F_\nu) = \theta \cap \eta$ if $\nu = (\eta, \theta, \beta)$, and by property (e) in Definition 2.1 we have that $\|F_\nu\|_p \approx |\eta|^{1/p-1/2}$, $0 < p \le \infty$. It is important that under certain conditions (see Remark 3.2) $\mathcal{F}_m$ is a stable basis:

Proposition 4.10. [13] *If $f \in W_m \cap L_p\,(\mathbb{R}^n)$, $0 < p \le \infty$ and $f = \sum_{\nu \in \mathrm{M}_m} a_\nu F_\nu$, then*

$$
\|f\|_p \sim \Big(\sum_{\nu\in \mathrm{M}_m} \|a_\nu F_\nu\|_p^p\Big)^{1/p} \sim 2^{m\left(\frac{1}{2}-\frac{1}{p}\right)} \Big(\sum_{\lambda \in \Lambda_m} |a_\nu|^p\Big)^{1/p}, \tag{4.11}
$$

with the obvious modification when $p = \infty$.

Let the coefficients $\left\{A^{\theta,\eta}_{\alpha,\beta}\right\}$ be determined from

$$
P_{\theta,\alpha} = \sum_{|\beta|<r} A^{\theta,\eta}_{\alpha,\beta} P_{\eta,\beta}, \quad \theta \in \Theta_m, \quad \eta \in \Theta_{m+1}. \tag{4.12}
$$

Let $\lambda \in \Lambda_m$ and $\lambda = (\theta, \alpha)$. Then using (4.12) and (4.10) we obtain the following *meshless two-scale relationship*

$$\begin{aligned}\varphi_\lambda = P_{\theta,\alpha}\tilde{\phi}_\theta &= \sum_{\eta\in\Theta_{m+1},\ \eta\cap\theta\neq\emptyset} P_{\theta,\alpha}(x)\,\tilde{\phi}_\theta\tilde{\phi}_\eta \\ &= \sum_{\eta\in\Theta_{m+1},\ \eta\cap\theta\neq\emptyset,\ |\beta|<r} A^{\theta,\eta}_{\alpha,\beta} P_{\eta,\beta}\tilde{\phi}_\theta\tilde{\phi}_\eta \ = \sum_{\eta\in\Theta_{m+1},\ \eta\cap\theta\neq\emptyset,\ |\beta|<r} A^{\theta,\eta}_{\alpha,\beta} F_{\eta,\theta,\beta},\end{aligned}$$

and hence $\varphi_\lambda \in W_m$. Also, if $\lambda \in \Lambda_{m+1}$ and $\lambda = (\eta, \beta)$, then

$$\varphi_\lambda = P_{\eta,\beta}\tilde{\phi}_\eta = \sum_{\theta\in\Theta_m:\theta\cap\eta\neq\emptyset} P_{\eta,\beta}\tilde{\phi}_\eta\tilde{\phi}_\theta = \sum_{\theta\in\Theta_m:\theta\cap\eta\neq\emptyset} F_{\eta,\theta,\beta}.$$

Combining the last two results we get that $\Phi_m \cup \Phi_{m+1} \subset W_m$. Recall that $S_m(x,y) = \sum_{\lambda\in\Lambda_m} \tilde{\varphi}_\lambda(y)\,\varphi_\lambda(x)$ and that $D_m(x,y) = S_{m+1}(x,y) - S_m(x,y)$. We use (3.9) and (4.12) to obtain

$$\begin{aligned}D_m(x,y) &= \sum_{\eta\in\Theta_{m+1}}\sum_{|\beta|<r} \tilde{\varphi}_{\eta,\beta}(y)\,P_{\eta,\beta}(x)\,\tilde{\phi}_\eta(x) - \sum_{\theta\in\Theta_m}\sum_{|\alpha|<r} \tilde{\varphi}_{\theta,\alpha}(y)\,P_{\theta,\alpha}(x)\,\tilde{\phi}_\theta(x) \\ &= \sum_{\theta\in\Theta_m}\sum_{\eta\in\Theta_{m+1}}\sum_{|\beta|<r} \tilde{\varphi}_{\eta,\beta}(y)\,P_{\eta,\beta}(x)\,\tilde{\phi}_\theta(x)\,\tilde{\phi}_\eta(x) \\ &\qquad - \sum_{\theta\in\Theta_m}\sum_{|\alpha|<r} \tilde{\varphi}_{\theta,\alpha}(y) \sum_{\eta\in\Theta_{m+1}}\sum_{|\beta|<r} A^{\theta,\eta}_{\alpha,\beta} P_{\eta,\beta}(x)\,\tilde{\phi}_\theta(x)\,\tilde{\phi}_\eta(x) \\ &= \sum_{\eta\in\Theta_{m+1}}\ \sum_{\theta\in\Theta_m:\theta\cap\eta\neq\emptyset}\ \sum_{|\beta|<r} \Big\{\tilde{\varphi}_{\eta,\beta}(y) - \sum_{|\alpha|<r} A^{\theta,\eta}_{\alpha,\beta}\tilde{\varphi}_{\theta,\alpha}(y)\Big\} P_{\eta,\beta}(x)\,\tilde{\phi}_\theta(x)\,\tilde{\phi}_\eta(x) \\ &=: \sum_{\nu\in\mathrm{M}_m} G_\nu(y)\,F_\nu(x),\end{aligned}$$

where $\{F_\nu\}$ are given by (4.10) and

$$G_\nu := G_{(\eta,\theta,\beta)} := \tilde{\varphi}_{\eta,\beta} - \sum_{|\alpha|<r} A^{\theta,\eta}_{\alpha,\beta}\tilde{\varphi}_{\theta,\alpha}. \tag{4.13}$$

Observe that for each $\nu = (\eta, \theta, \beta) \in \mathrm{M}_m$, since $\theta \cap \eta \neq \emptyset$, then (3.18) implies that the dual G_ν has fast decay with respect to the quasi-distance induced by the cover. Thus we obtain the two level split representation for the wavelet operators

$$D_m(f) = \sum_{\nu\in\mathrm{M}_m} \langle f, G_\nu\rangle\, F_\nu, \quad m \in \mathbb{Z}, \tag{4.14}$$

This also yields a representation for the elements of our discrete wavelet frame (4.5)

$$D_m(x, y_{m+N,k}) = \sum_{\nu\in\mathrm{M}_{m+1}} G_\nu(y_{m+N,k})\,F_\nu(x),$$

and thus implies that

$$\overline{span}\,\{D_m(x, y_{m+N,k}) \ :\ k \in I_{m+N}\} \subseteq W_m.$$

We conclude with the main result of this subsection.

Corollary 4.11. *The two level splits* $\{G_\nu\}$ *defined in* (4.13) *are a frame* (*see Definition 1.1*).

Proof. Let $f \in L_2(\mathbb{R}^n)$. Since $I = \sum_m D_m$, we have by (4.14)

$$f = \sum_m D_m(f) = \sum_m \sum_{\nu \in \mathrm{M}_m} \langle f, G_\nu \rangle F_\nu .$$

We combine Proposition 4.6 with Proposition 4.10

$$\|f\|_2^2 \sim \sum_m \int_{\mathbb{R}^n} |D_m(f)(x)|^2 \, dx \sim \sum_m \sum_{\nu \in \mathrm{M}_m} \|\langle f, G_\nu \rangle F_\nu\|_2^2 \sim \sum_m \sum_{\nu \in \mathrm{M}_m} |\langle f, G_\nu \rangle|^2 .$$

$\diamondsuit$

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